Lecture 30

Greedy: MST, Kruskal's Algorithm

Source: Introduction to Algorithms, CLRS

Cut Connection of MST

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Let’s amend the previous lemma so that it can help us in building an MST!

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Proof:

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Proof: Similar to the previous proof.

Kruskal's Algorithm: Demonstration

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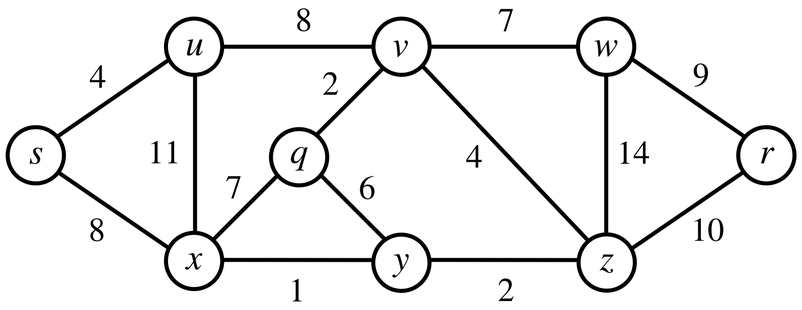

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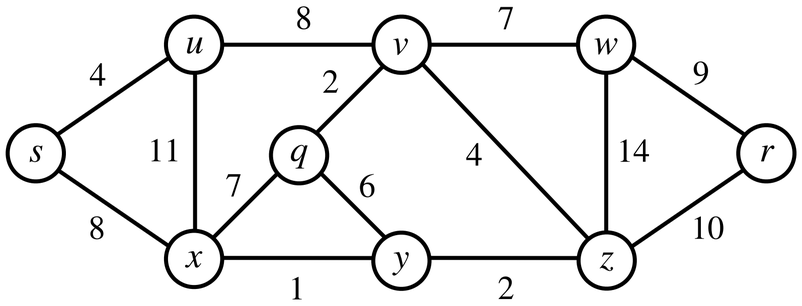

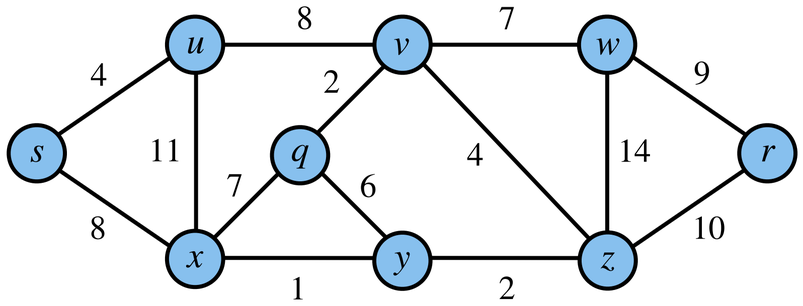

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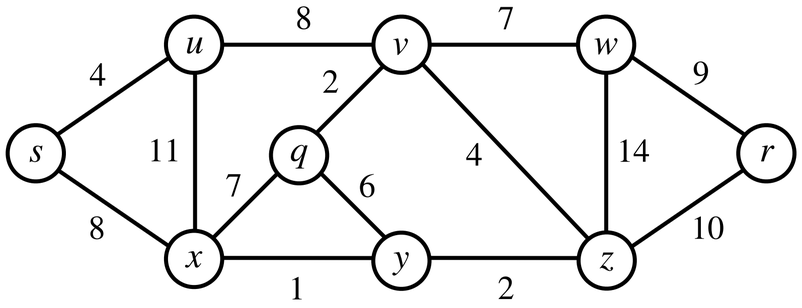

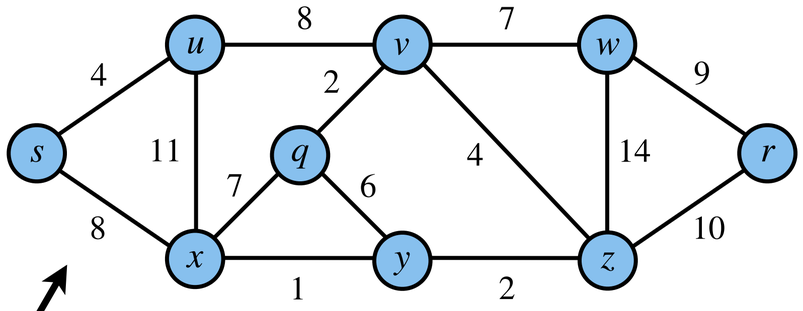

Start with single node subtrees.

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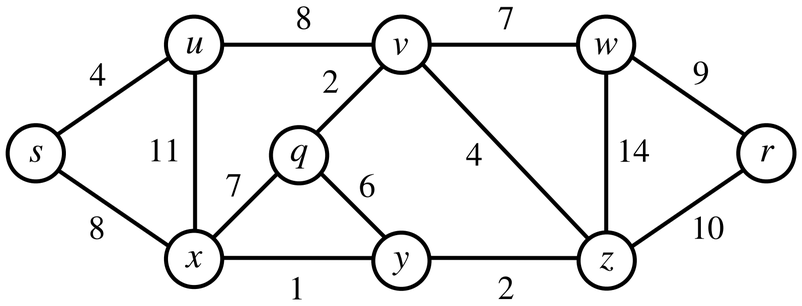

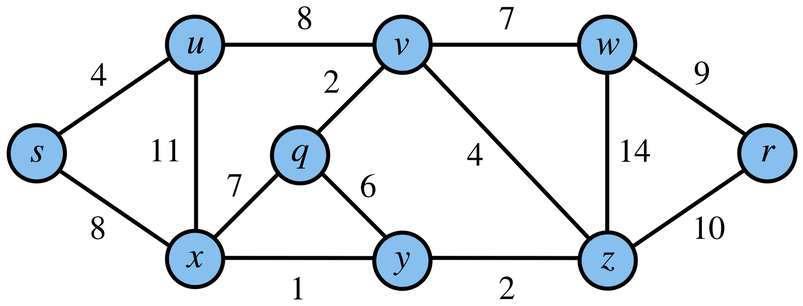

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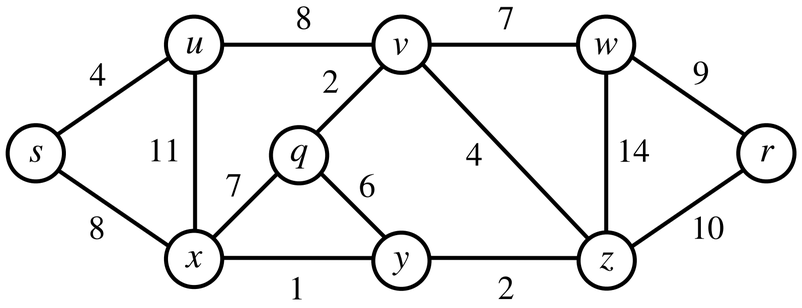

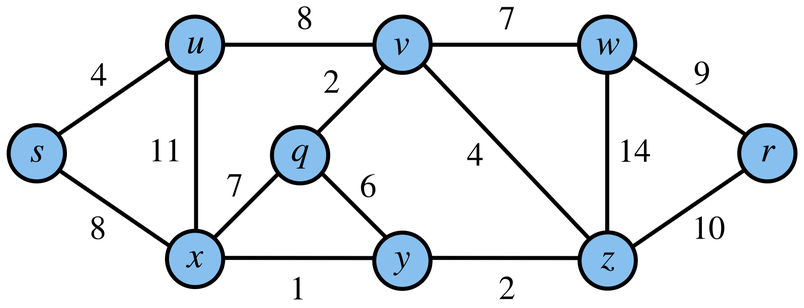

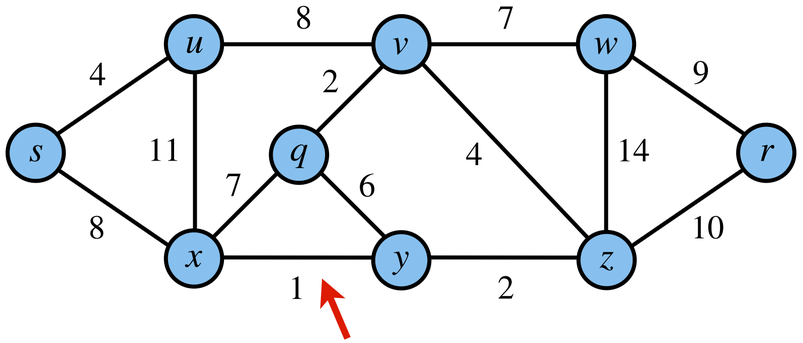

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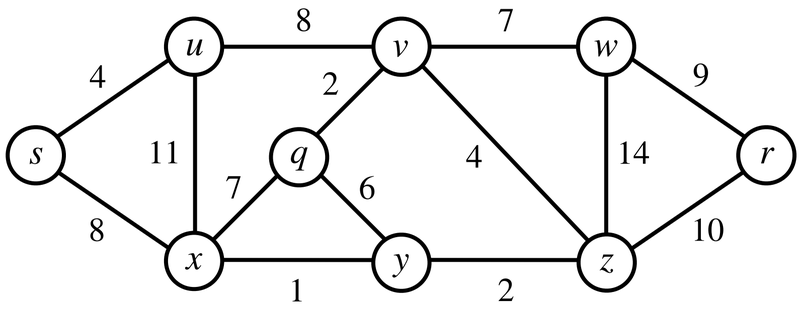

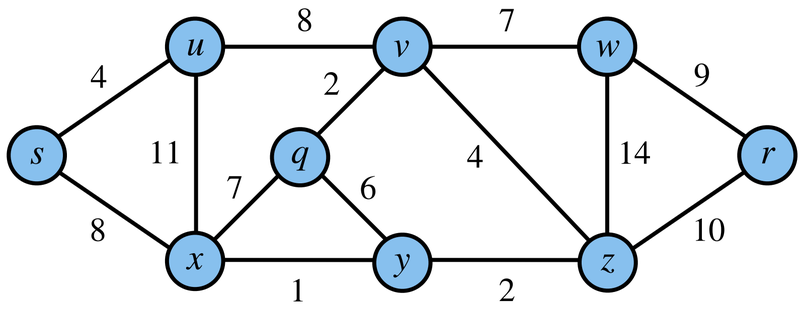

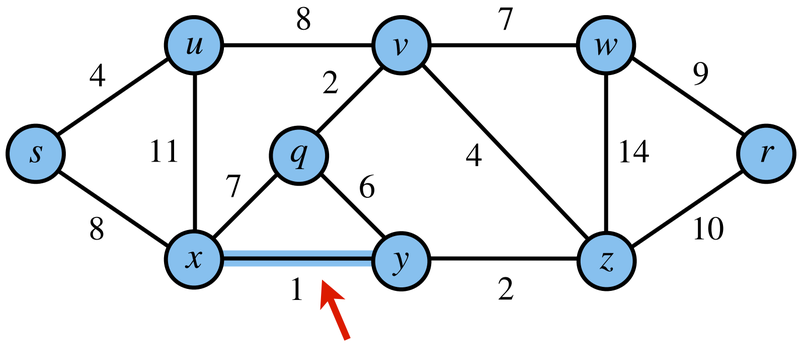

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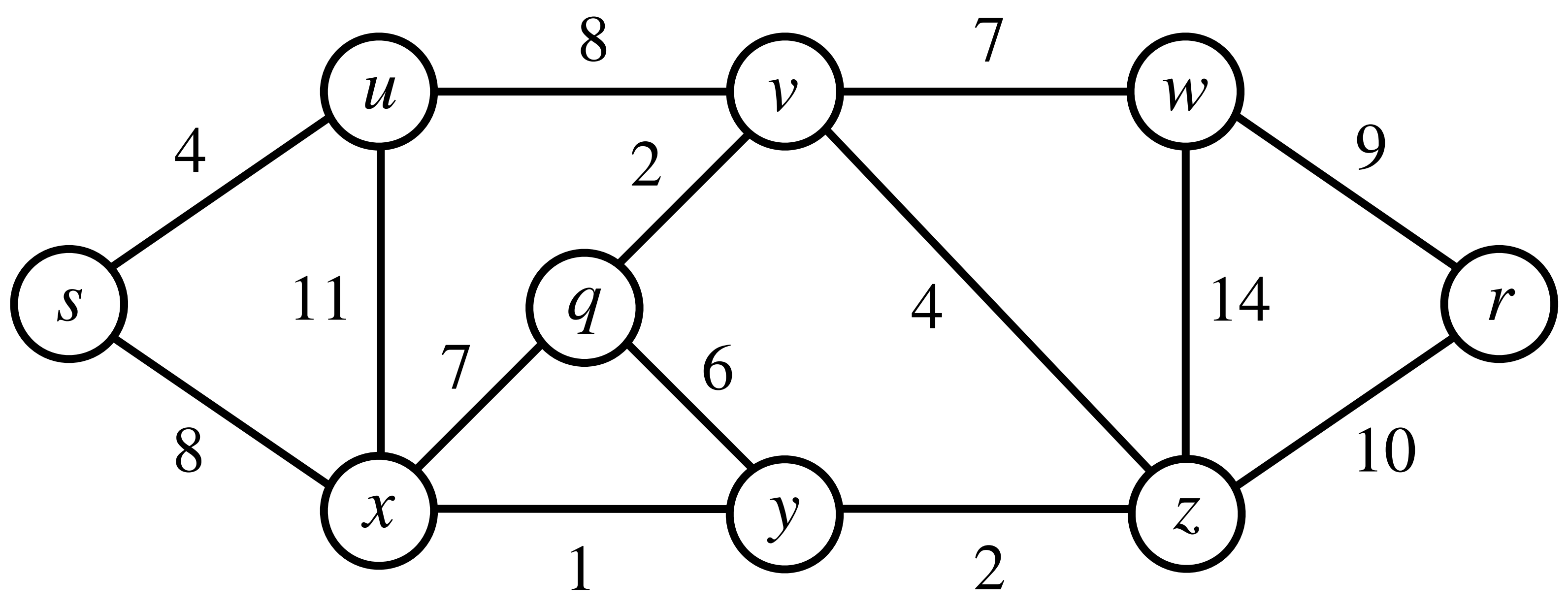

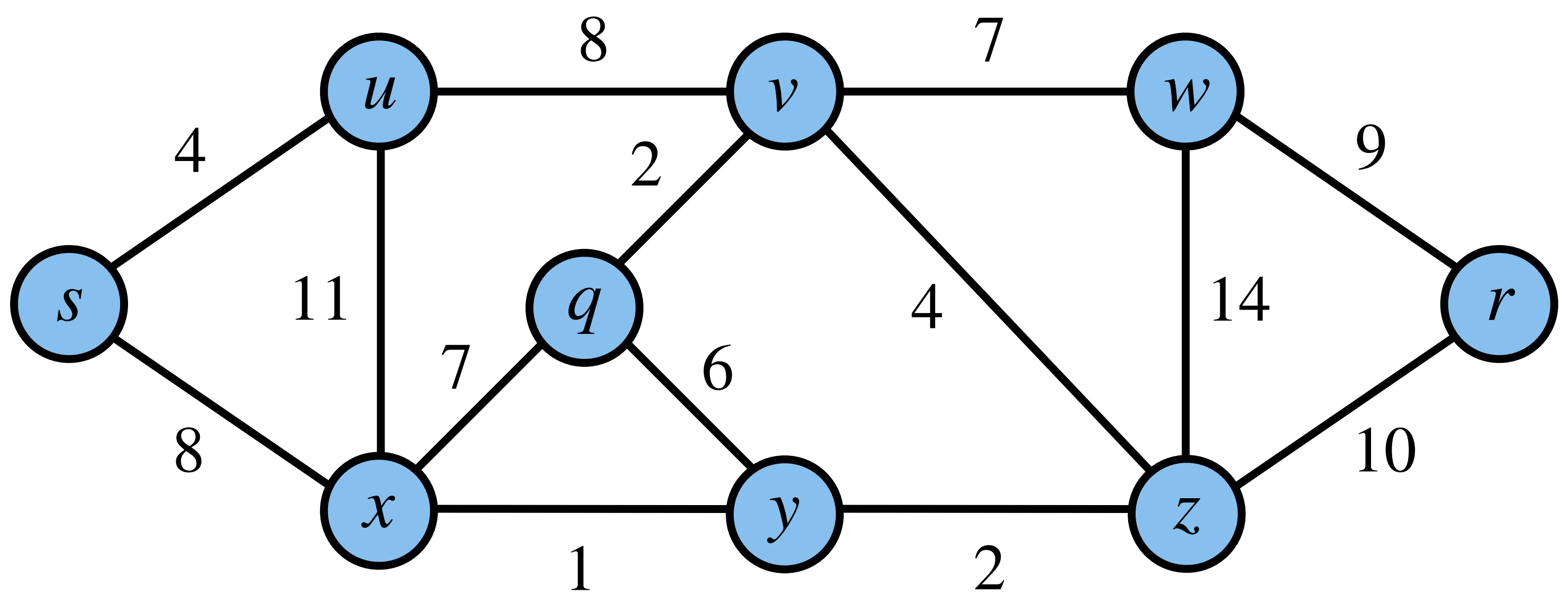

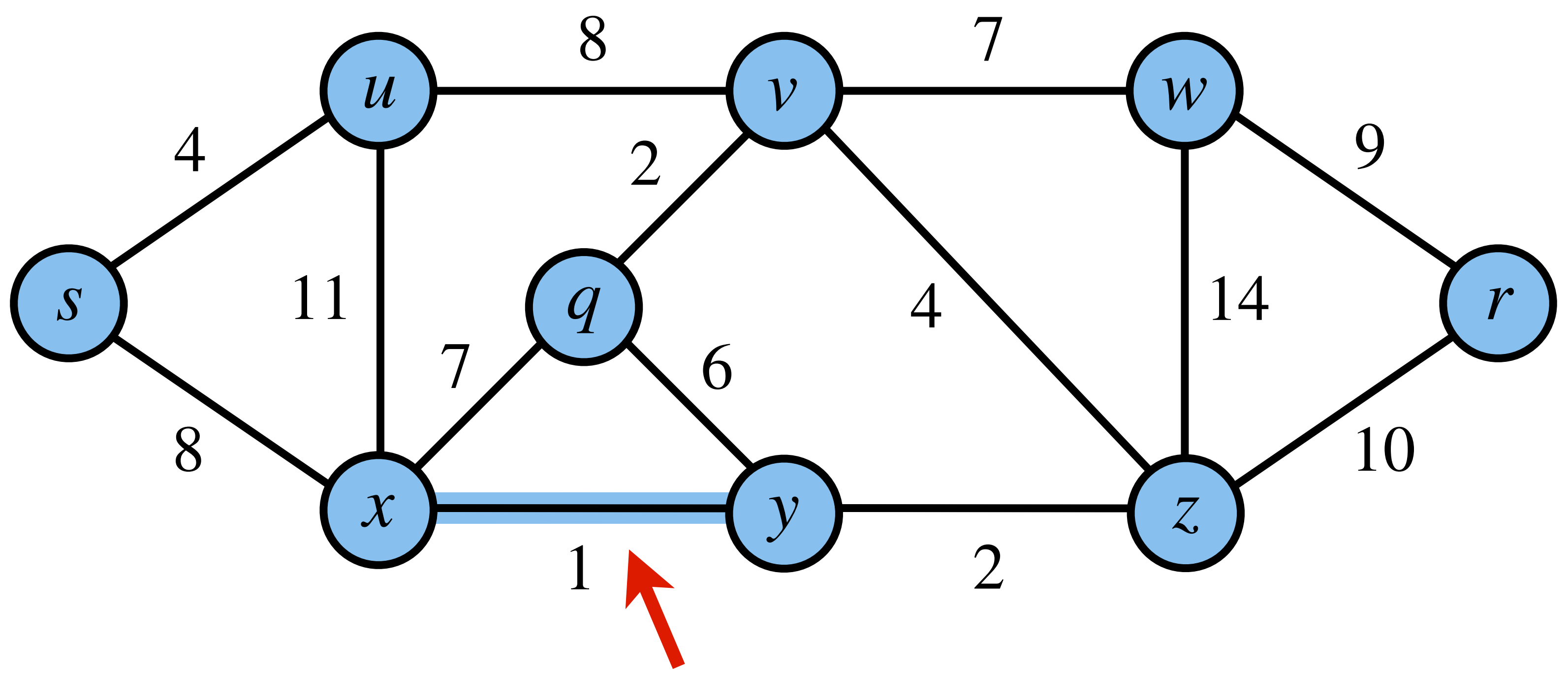

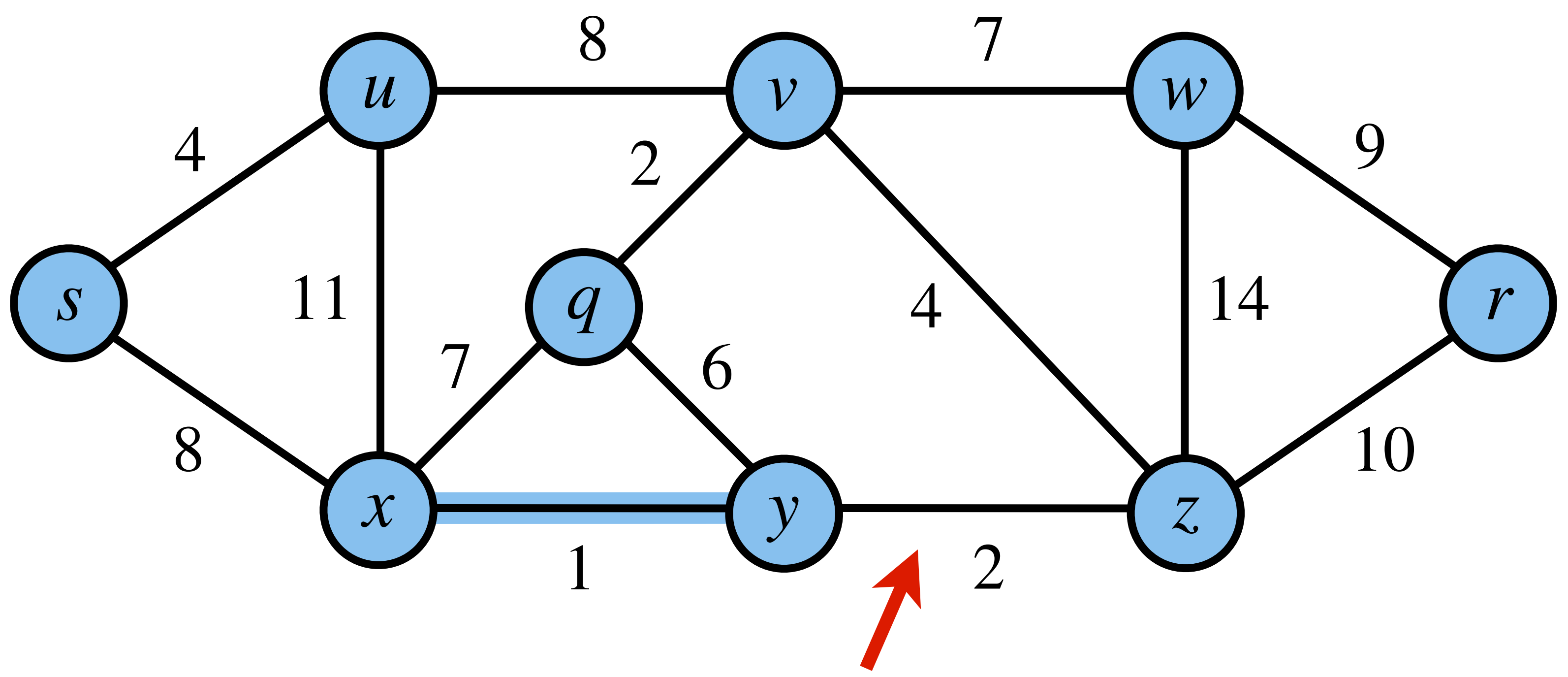

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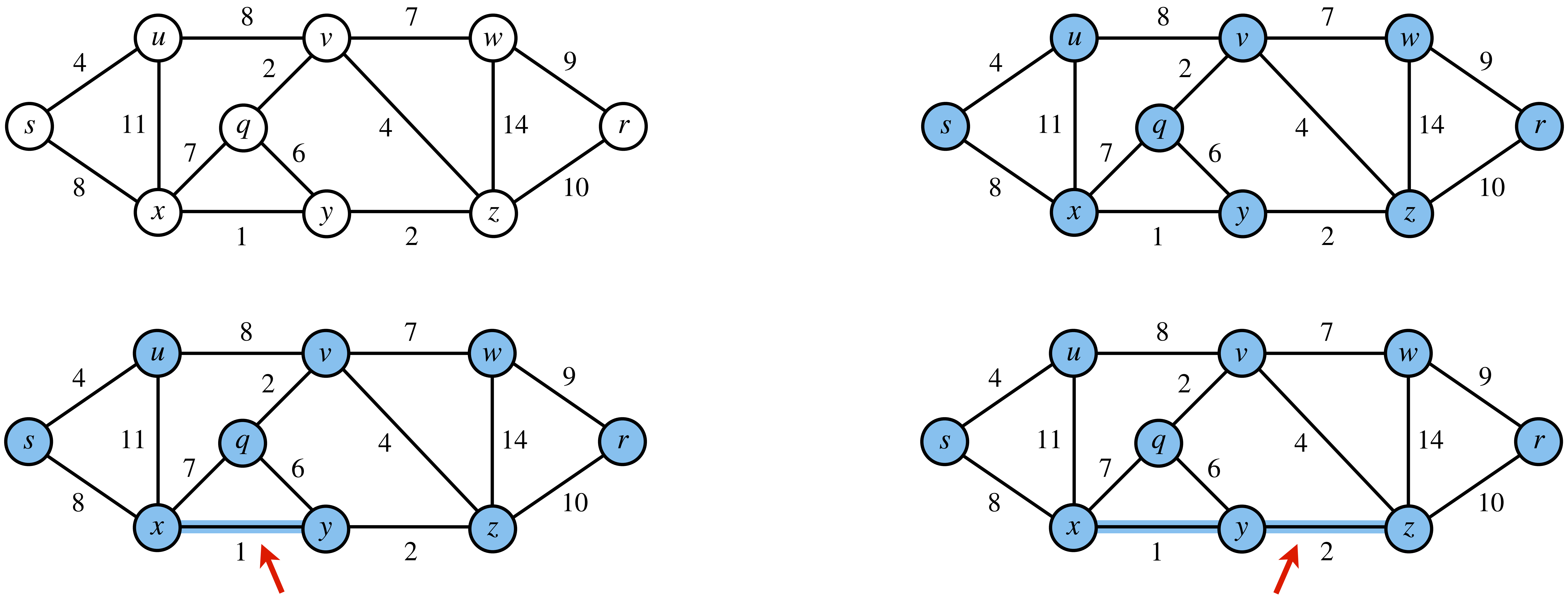

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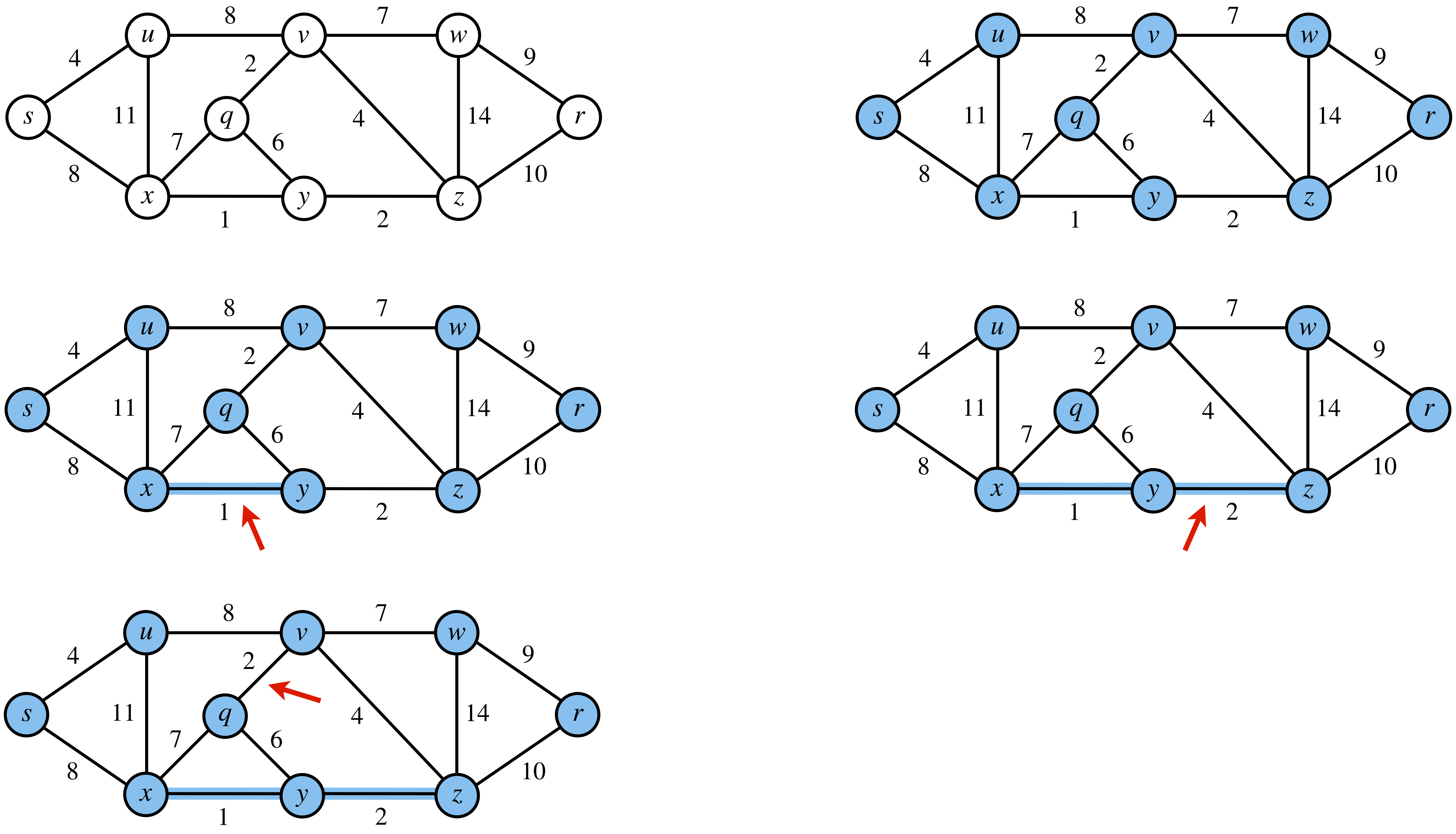

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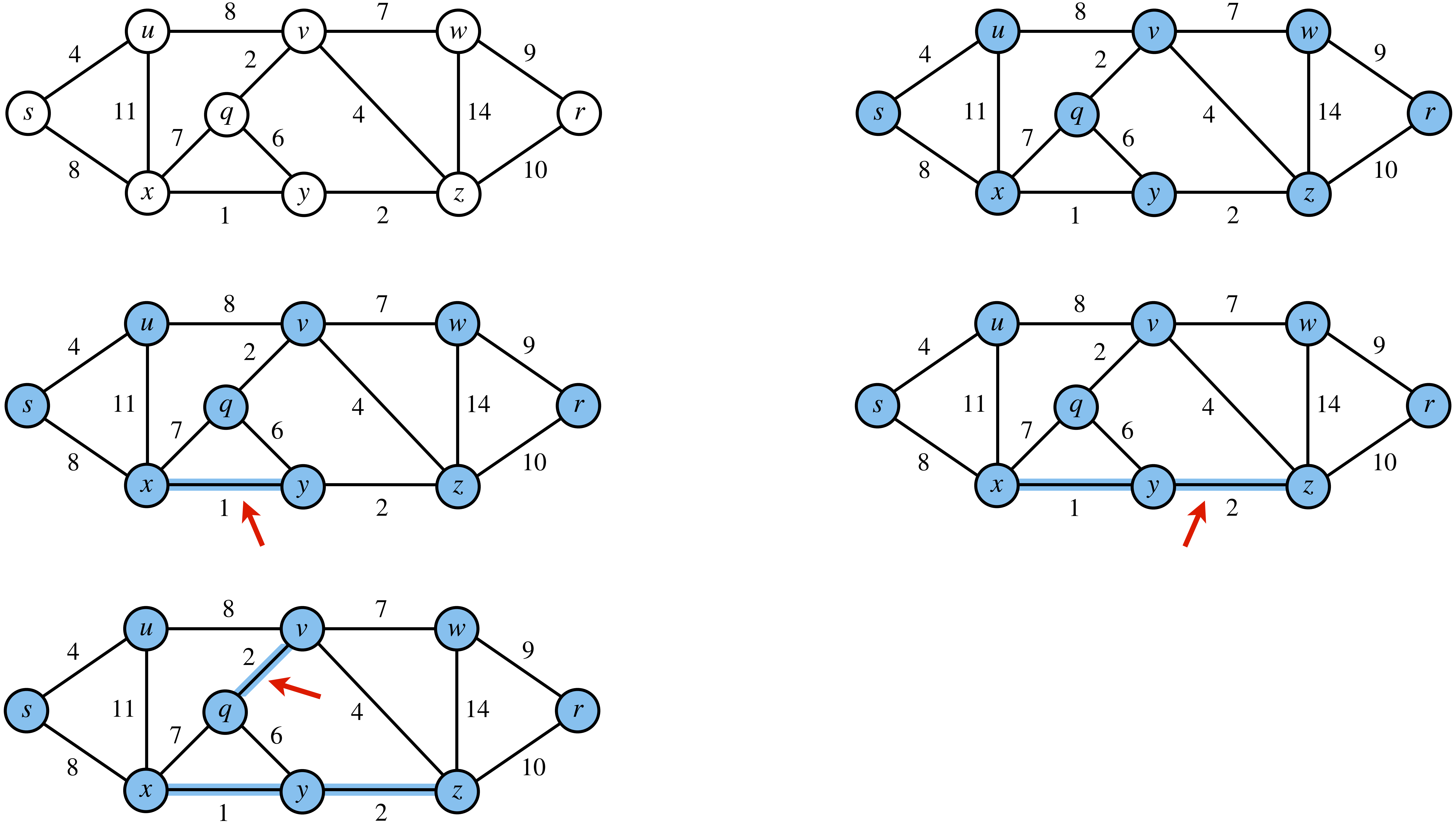

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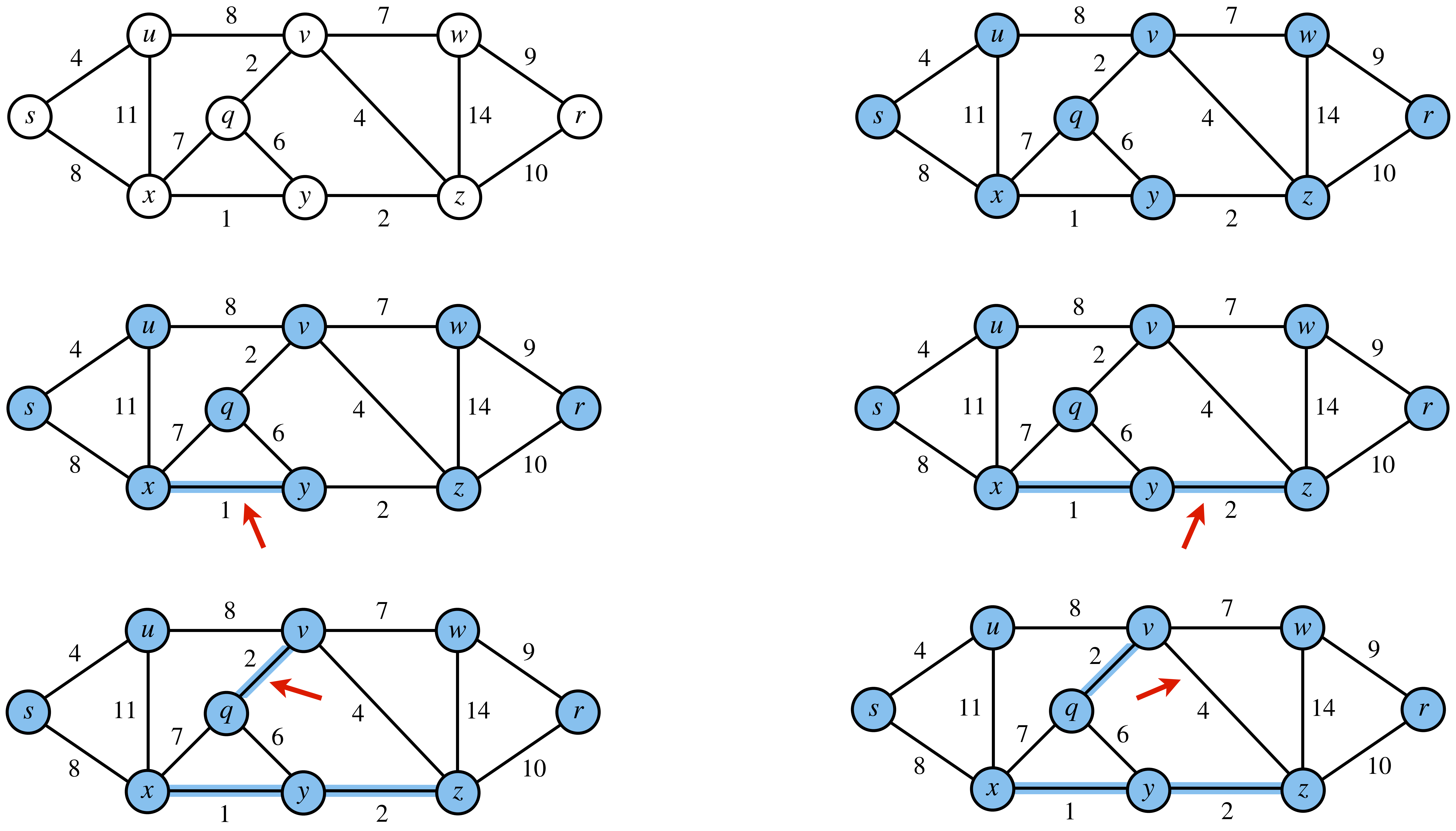

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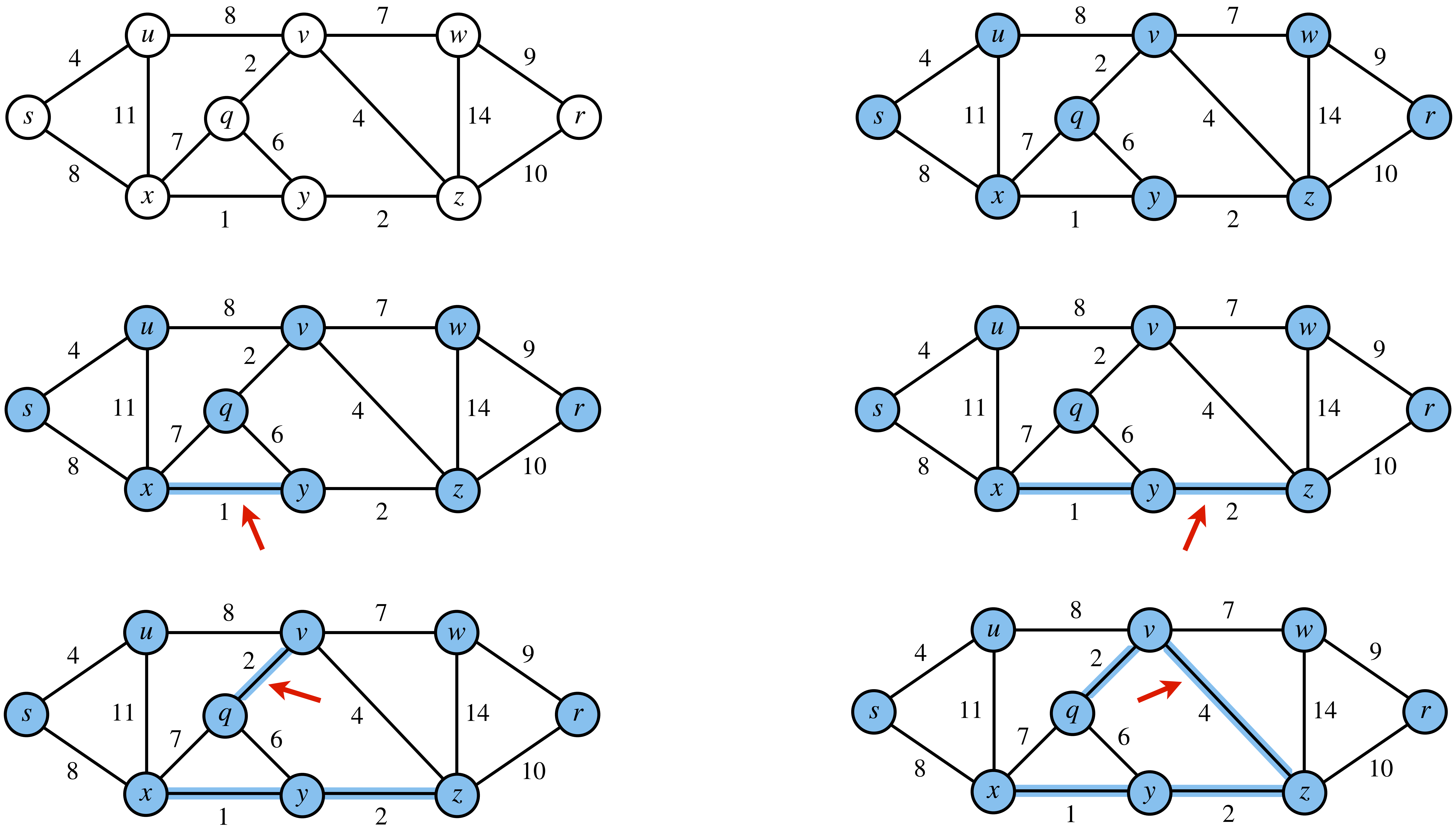

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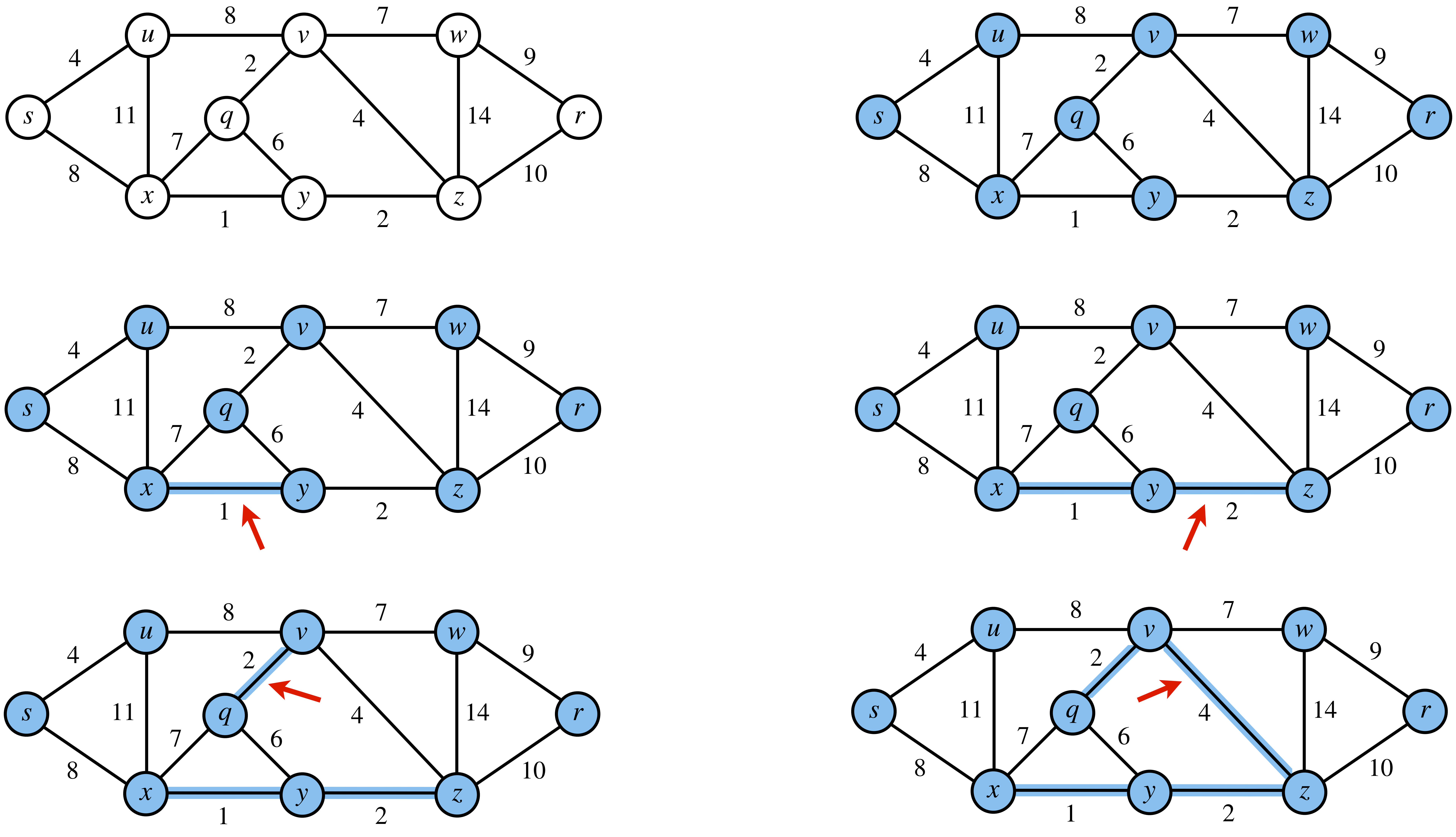

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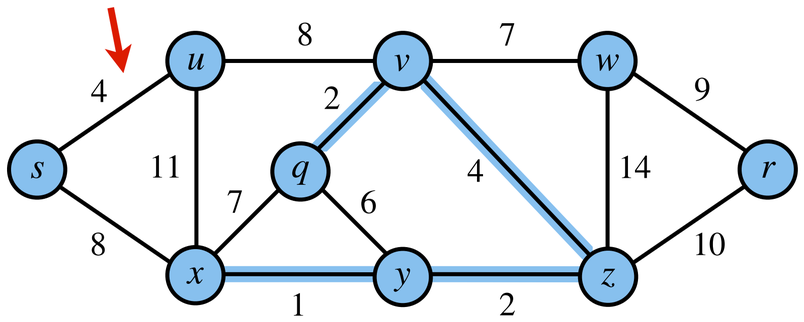

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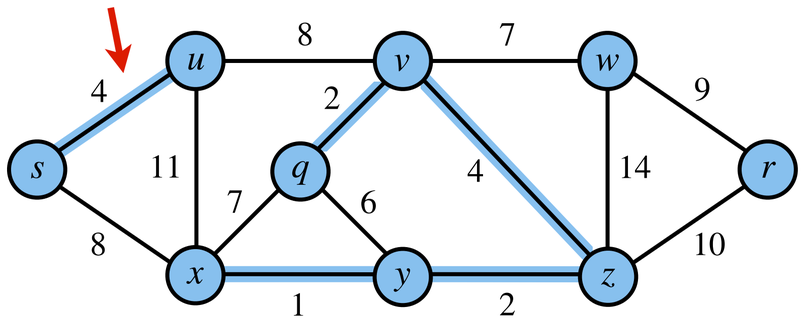

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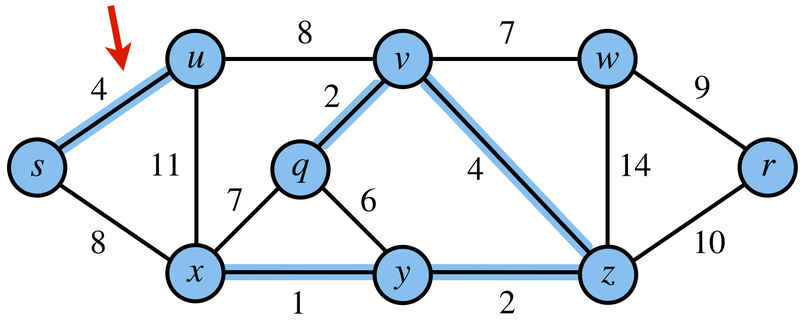

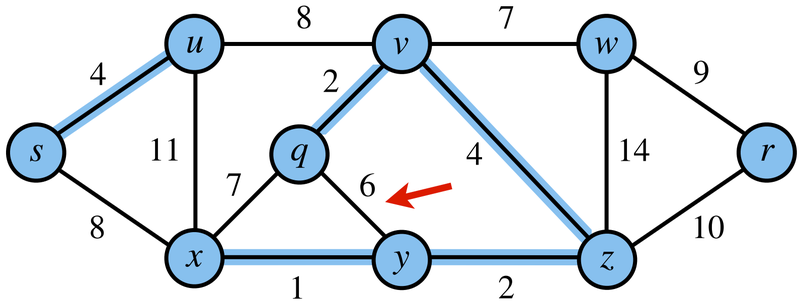

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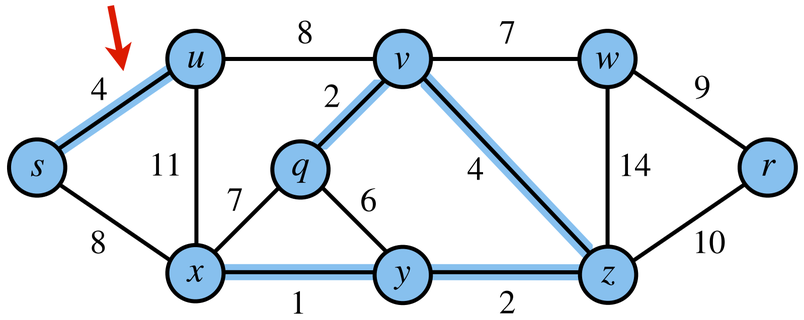

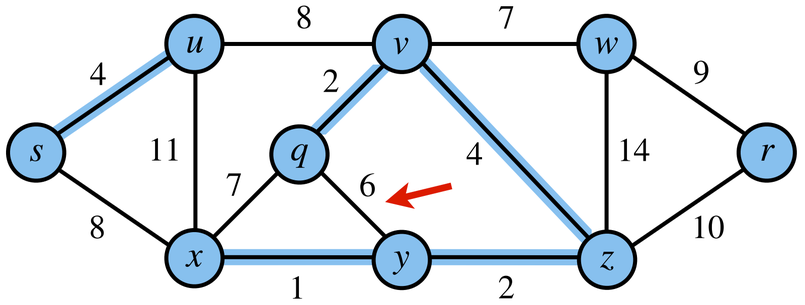

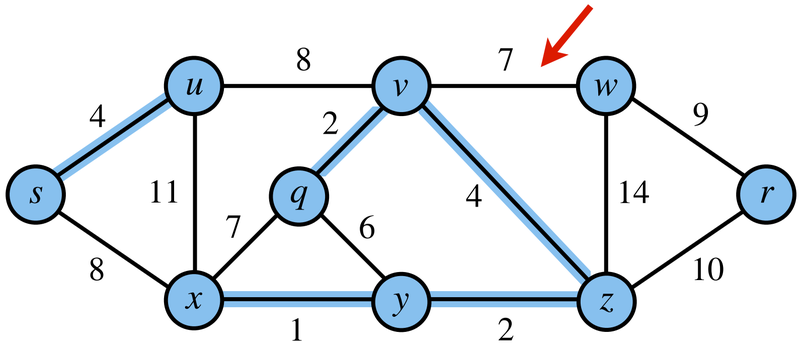

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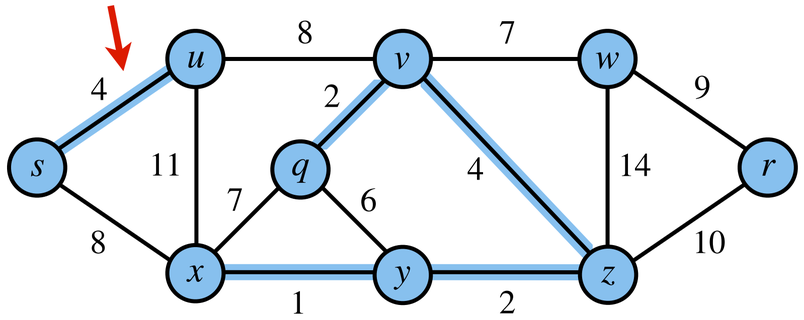

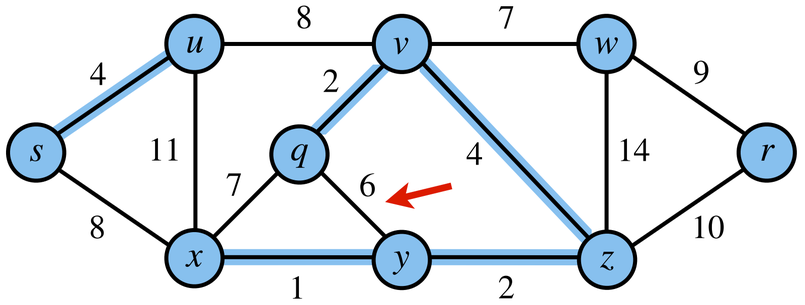

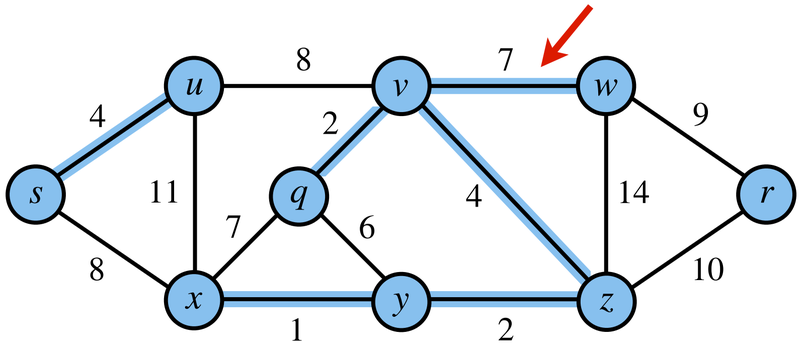

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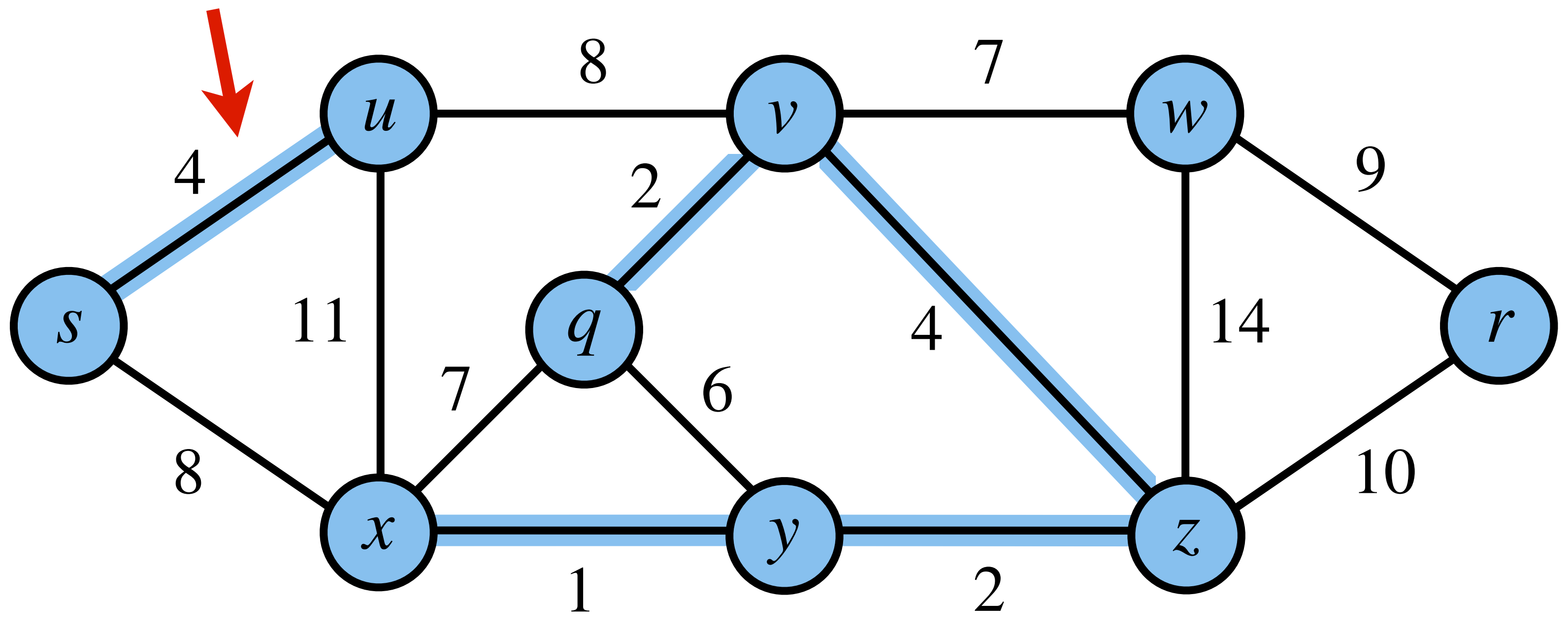

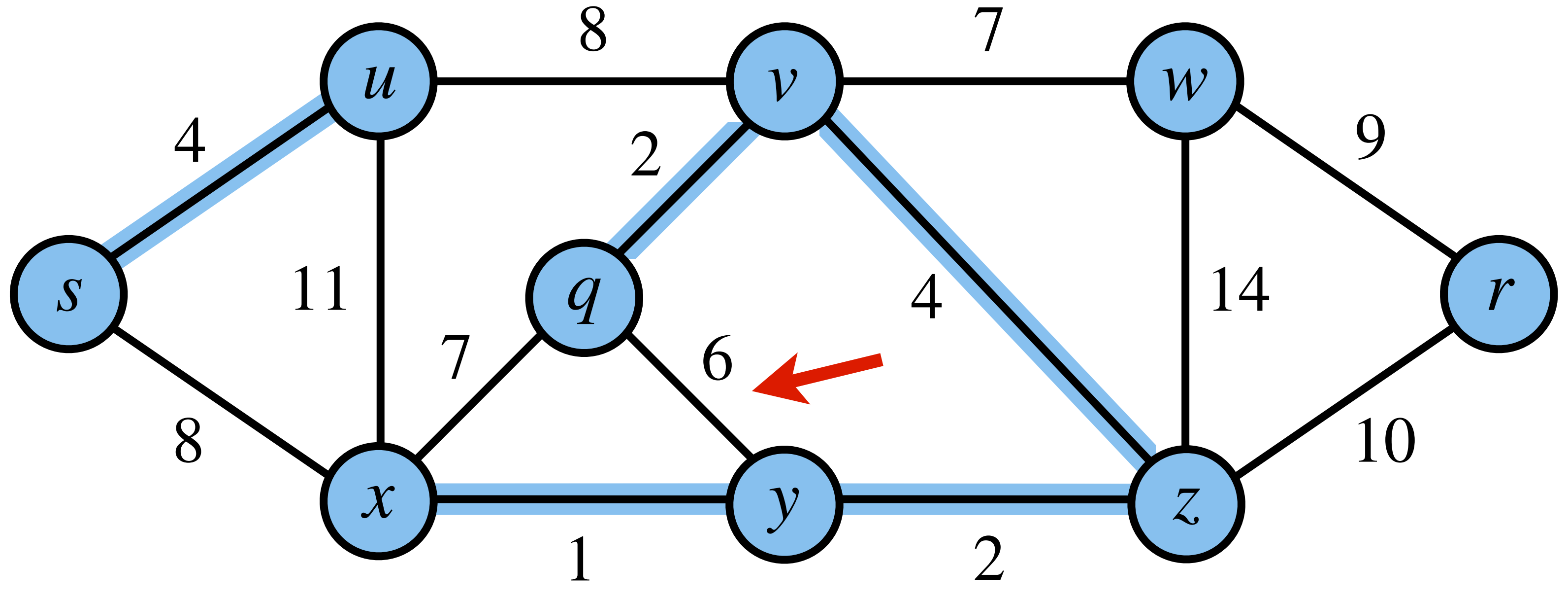

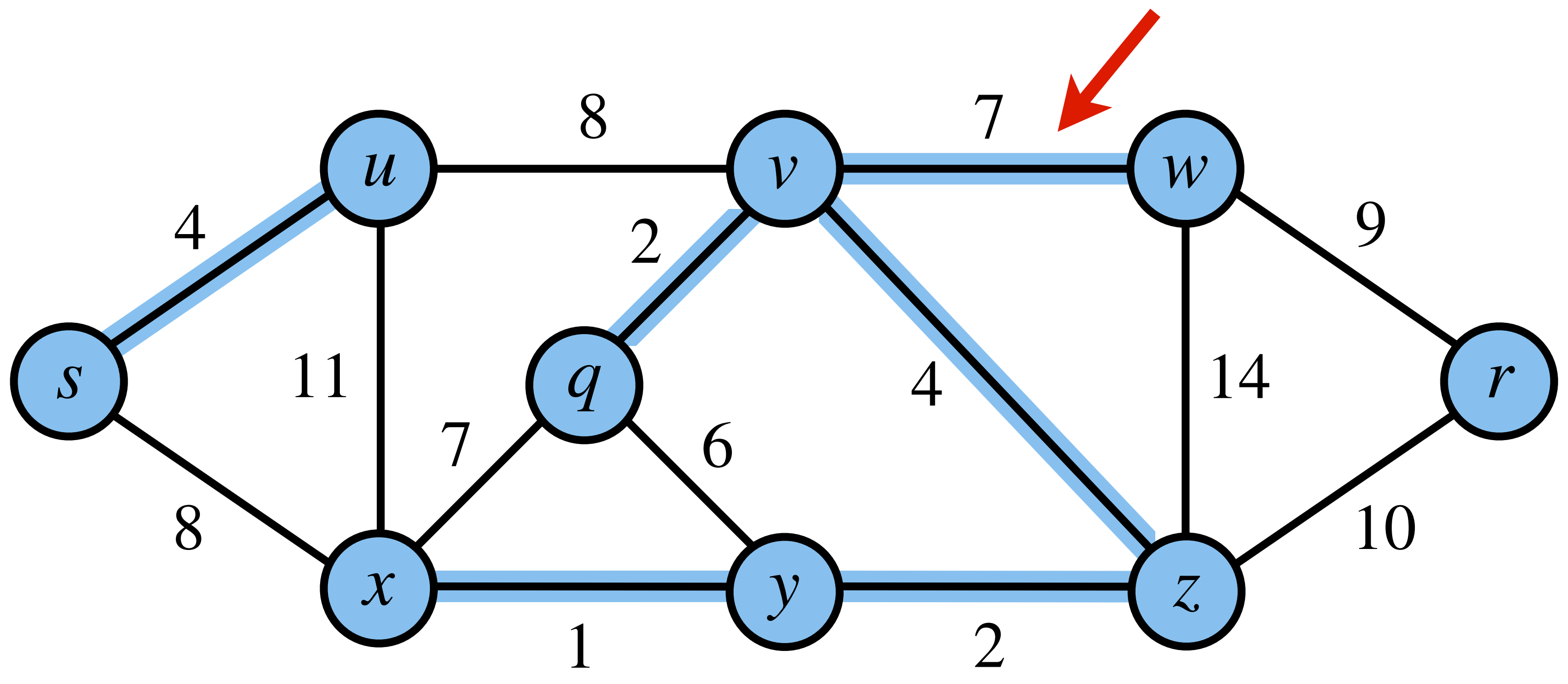

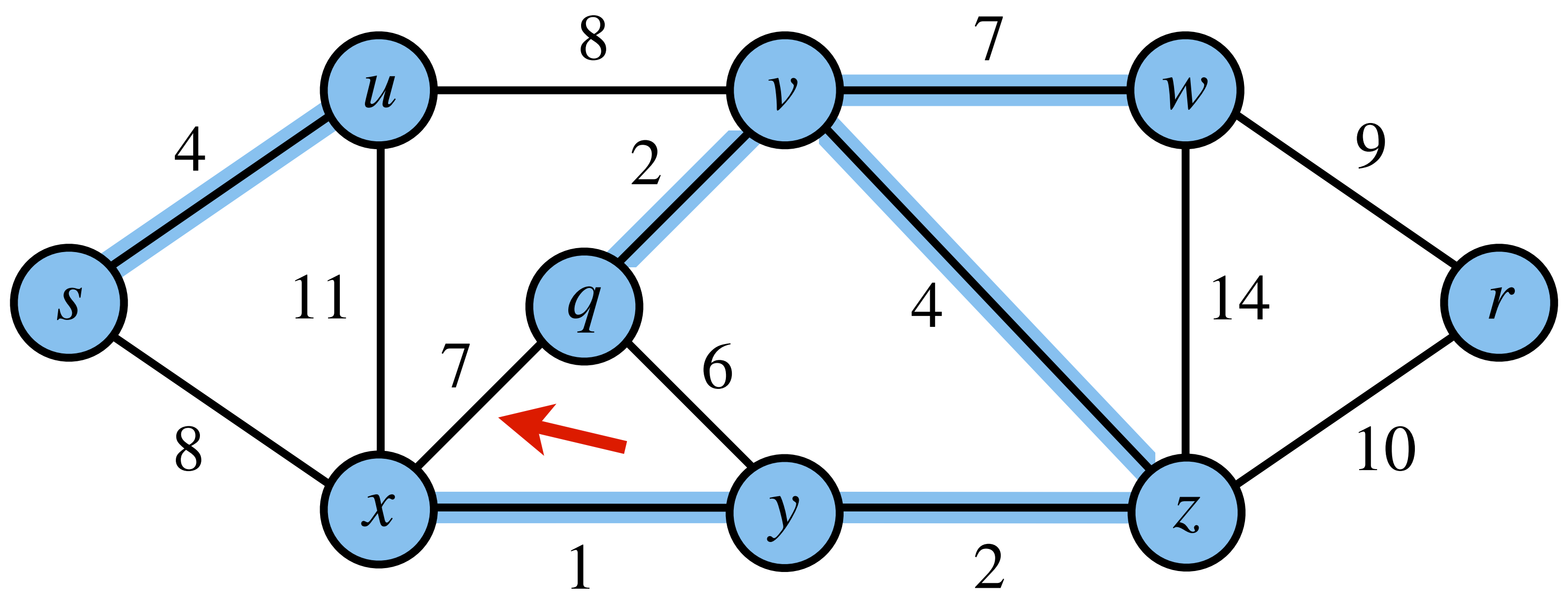

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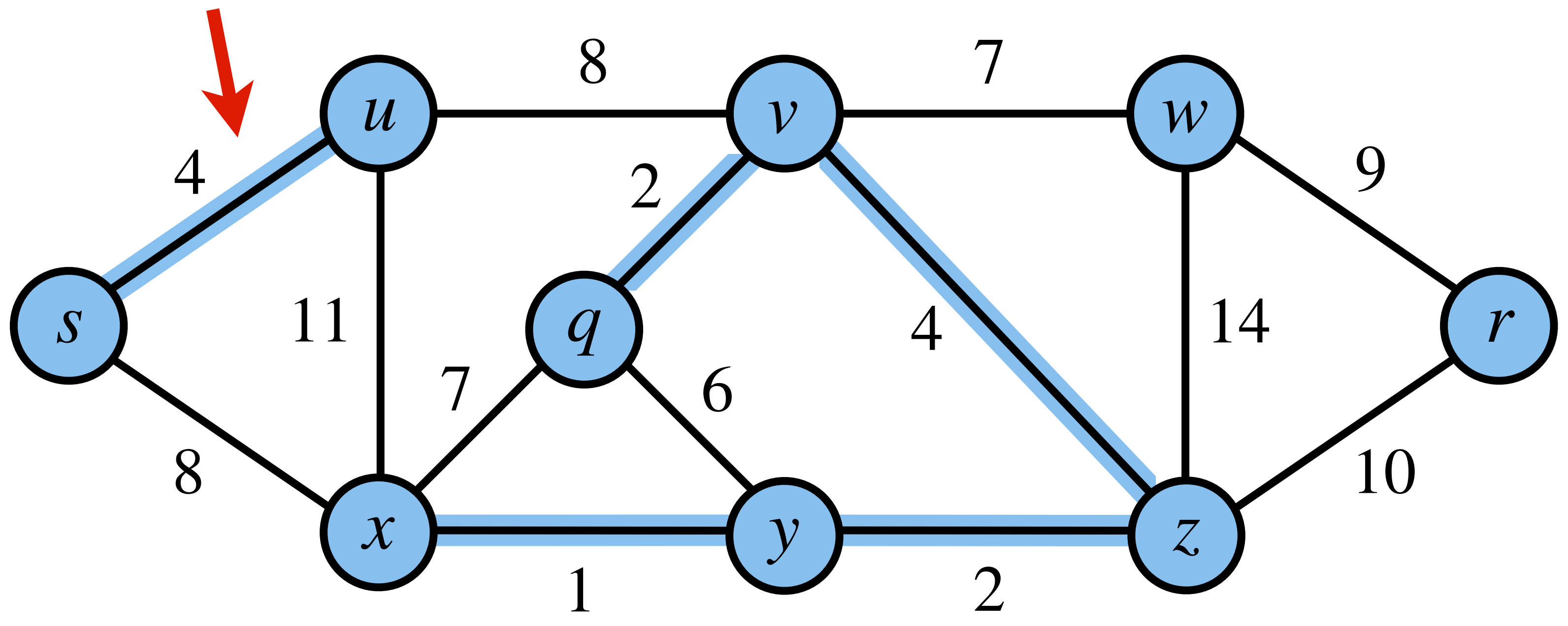

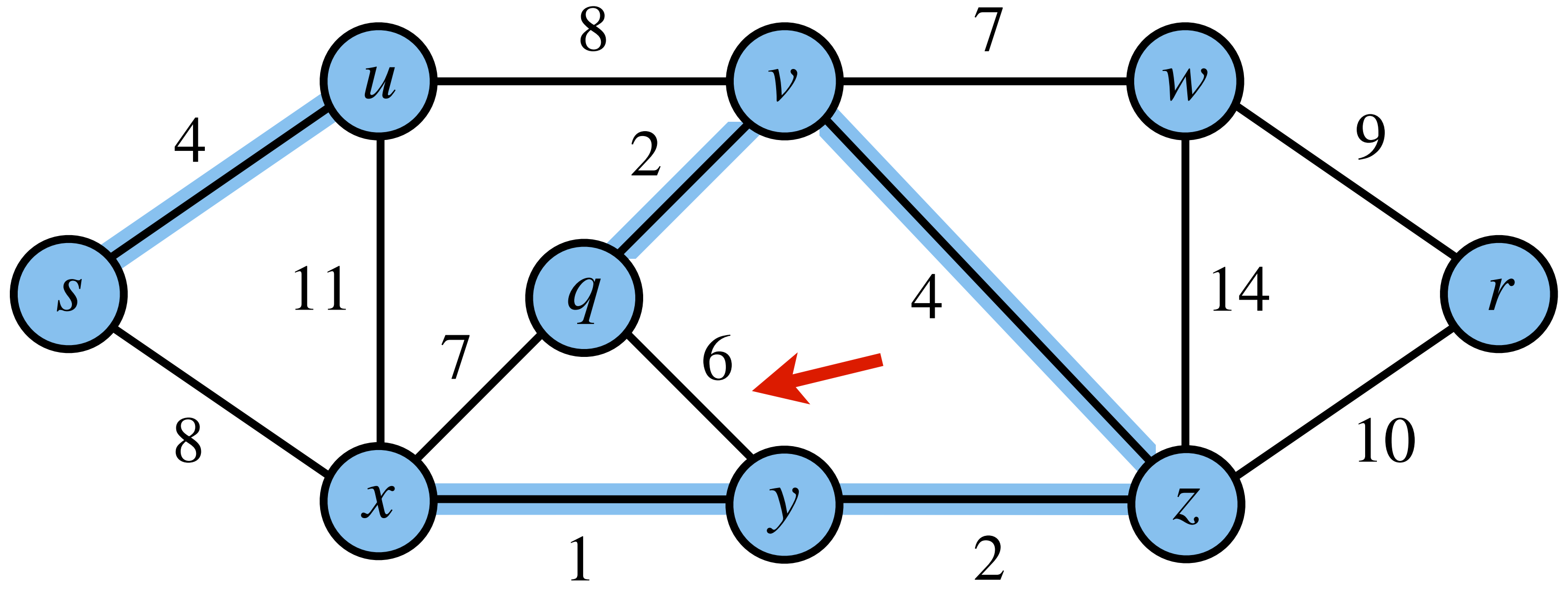

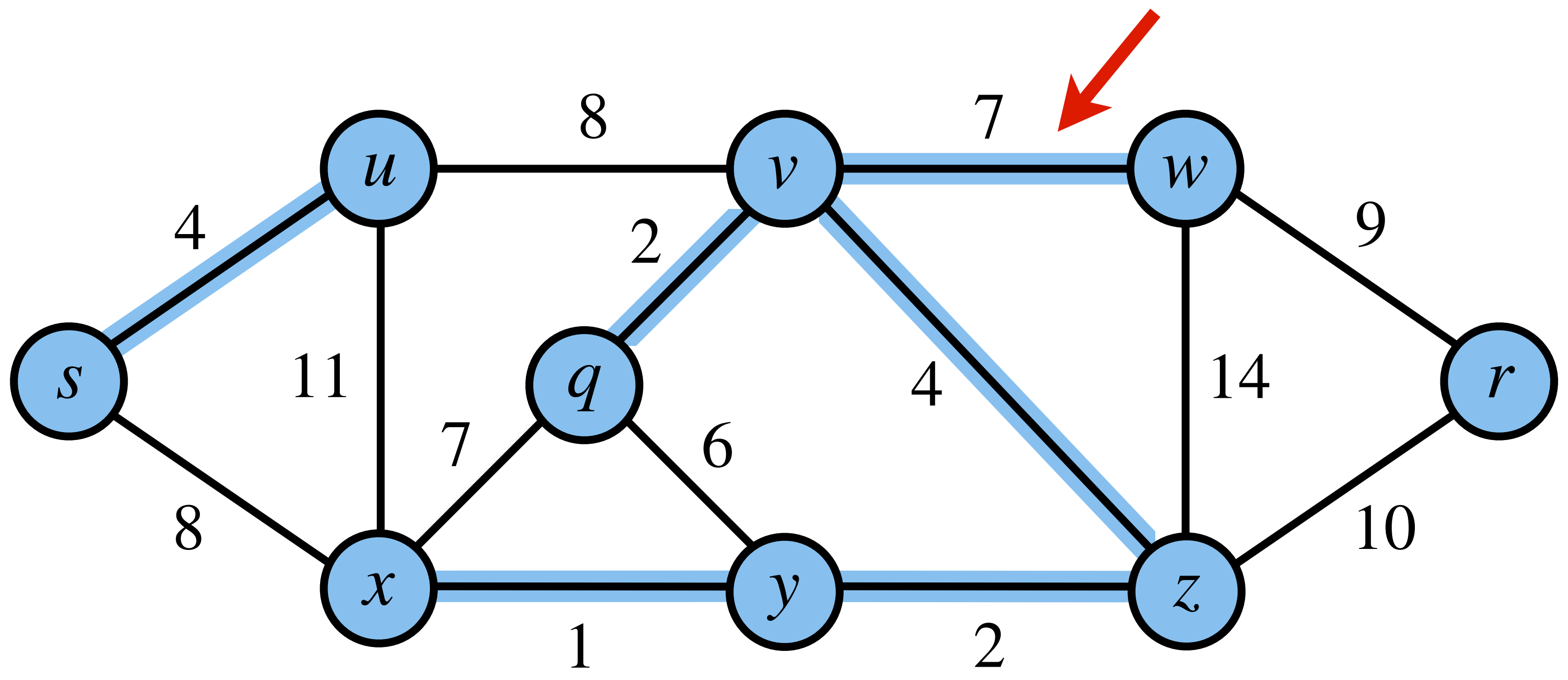

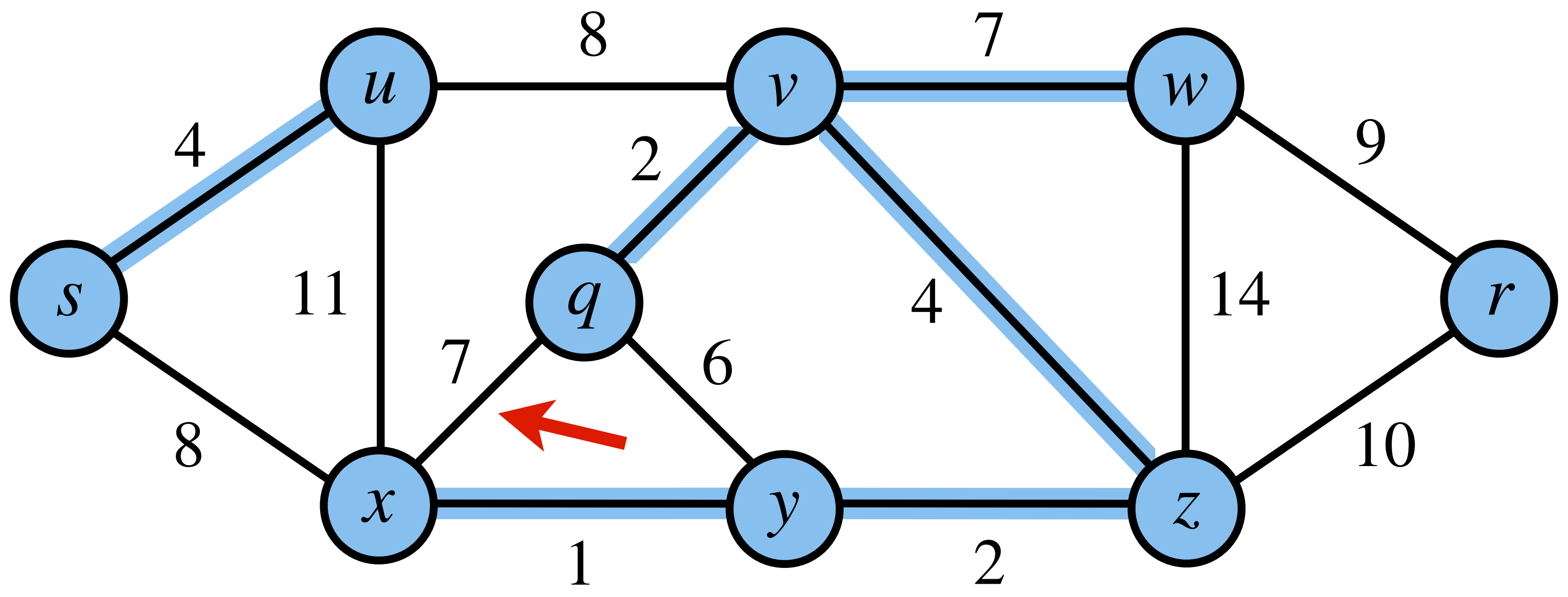

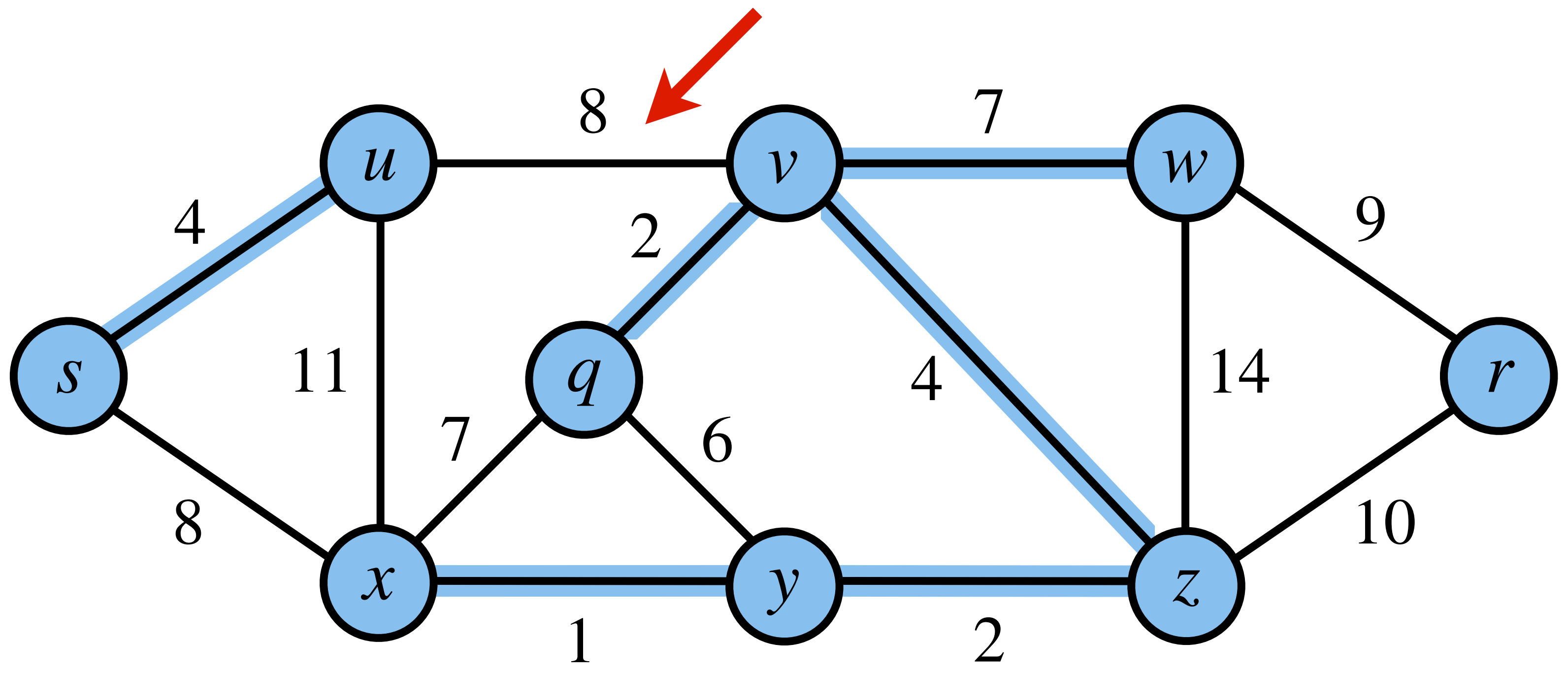

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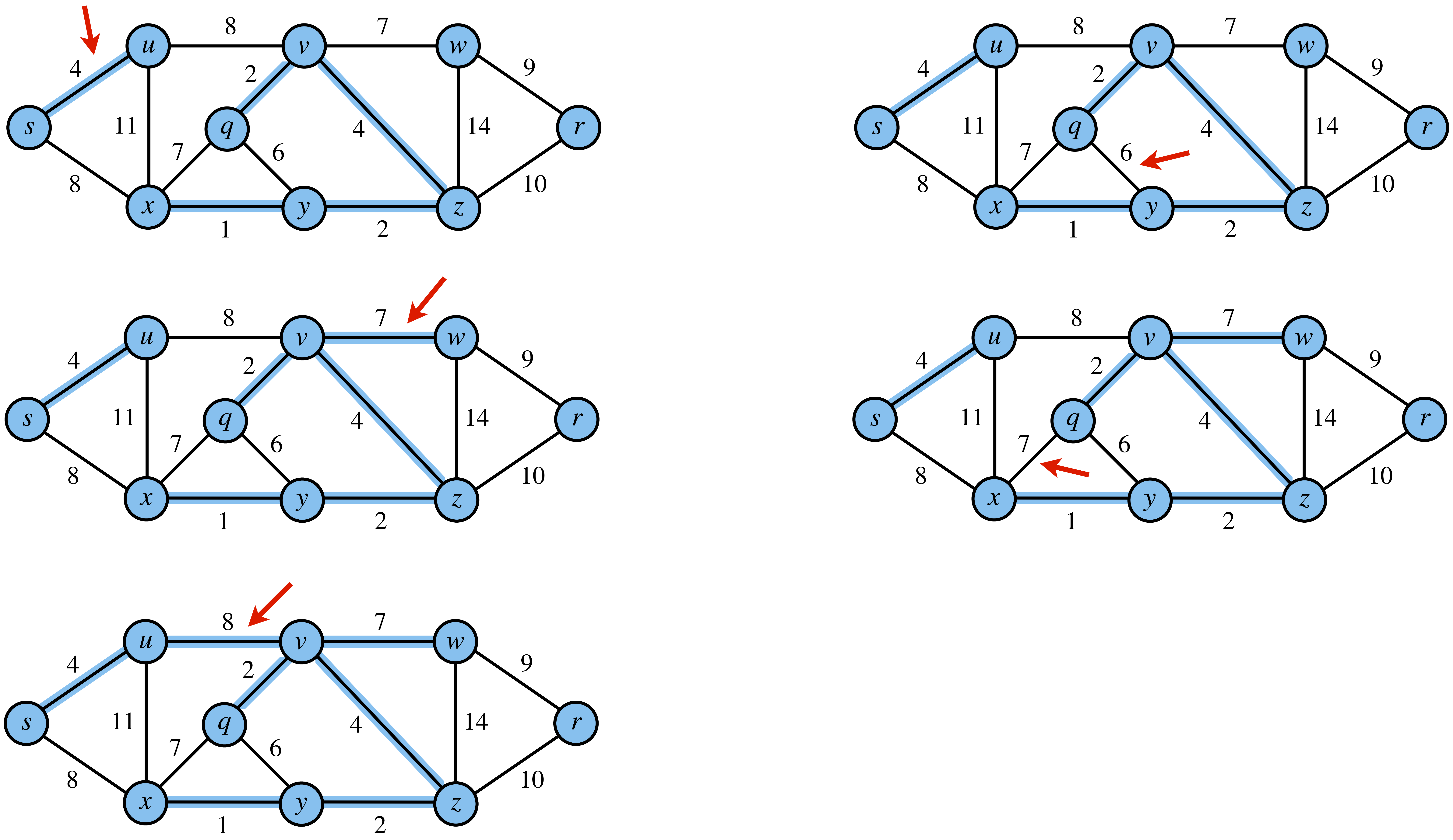

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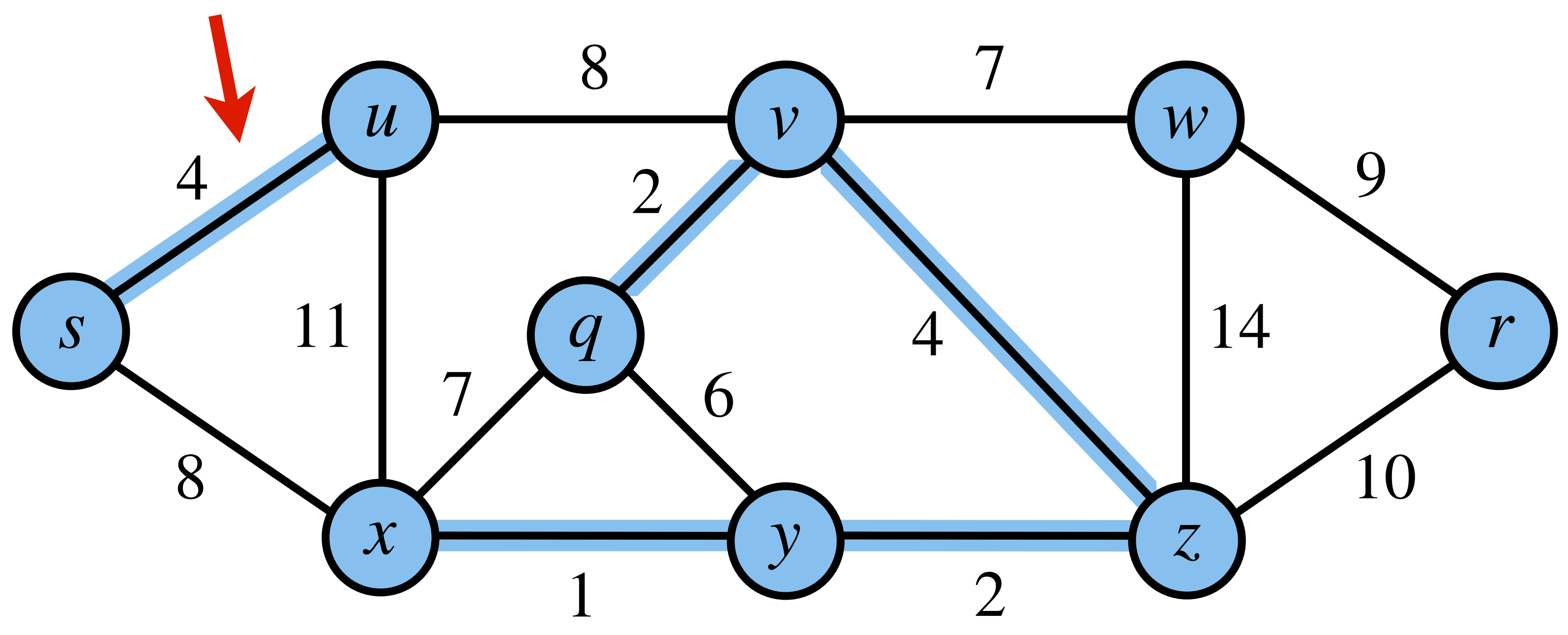

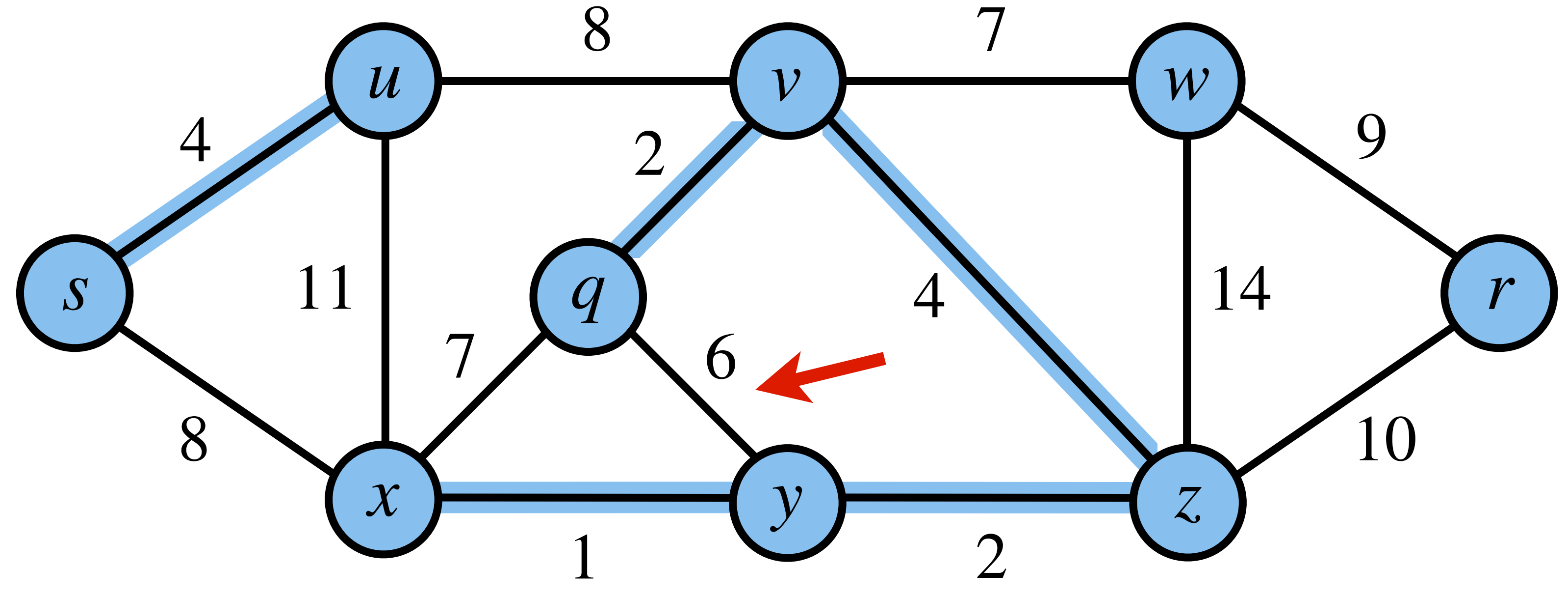

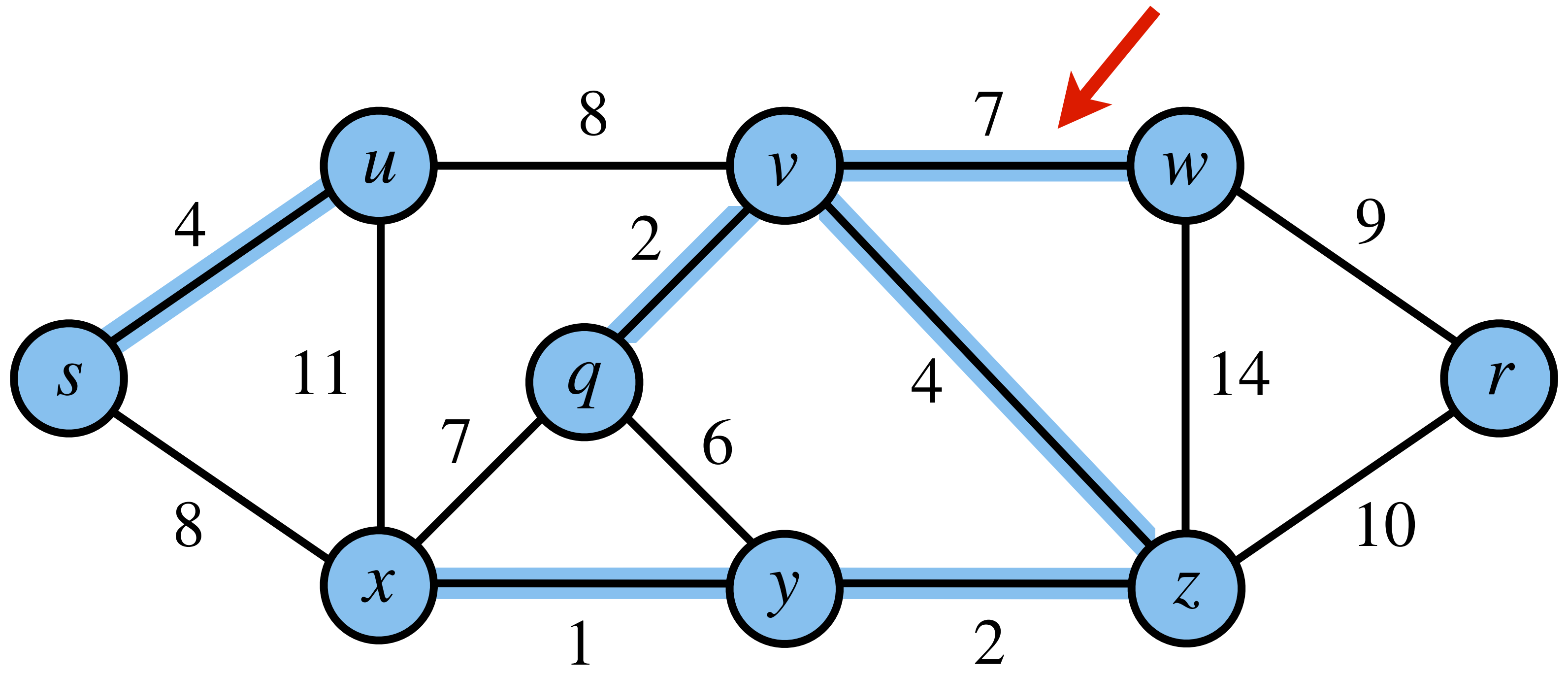

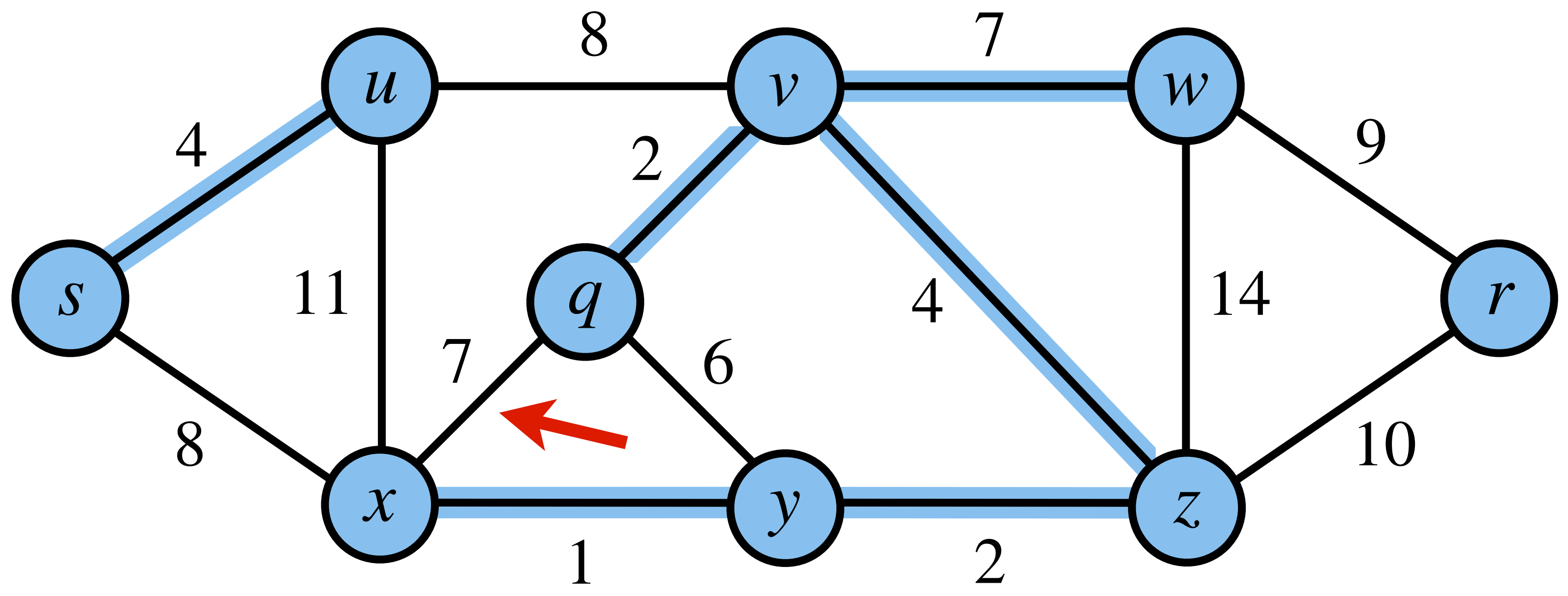

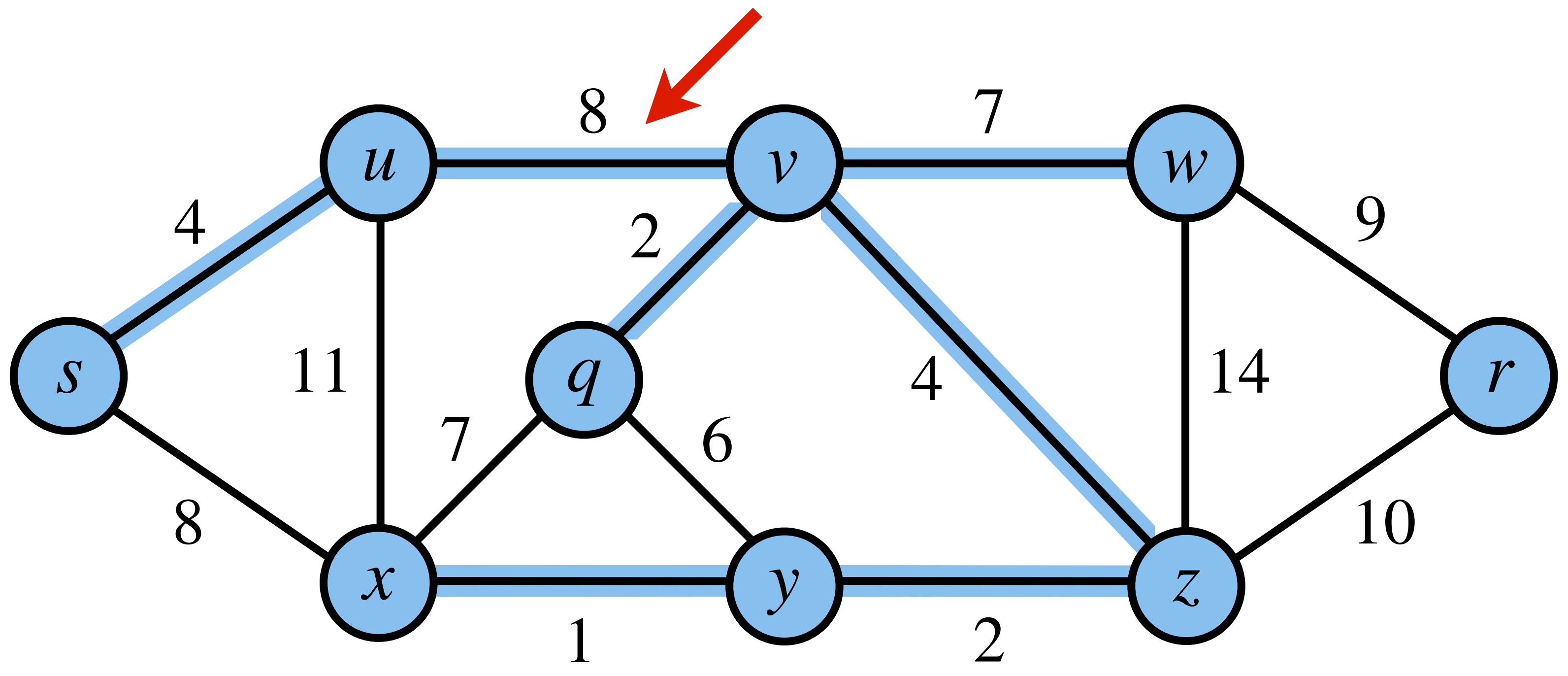

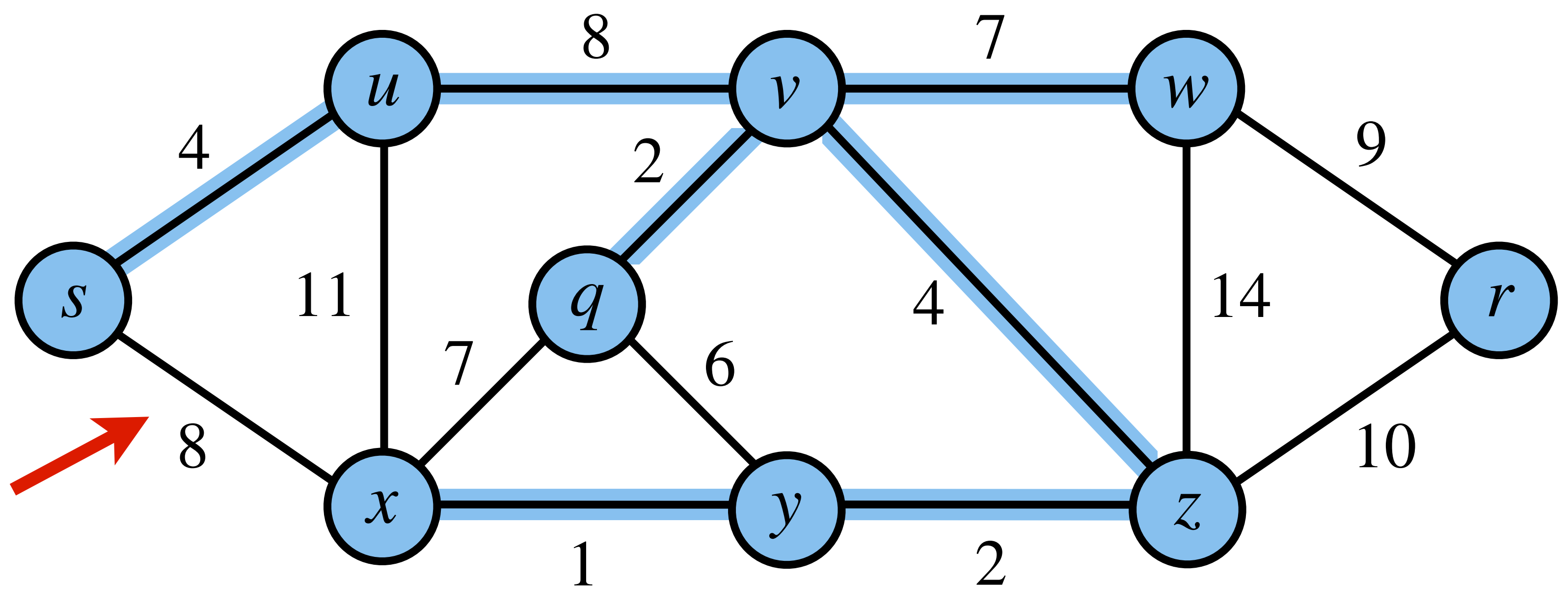

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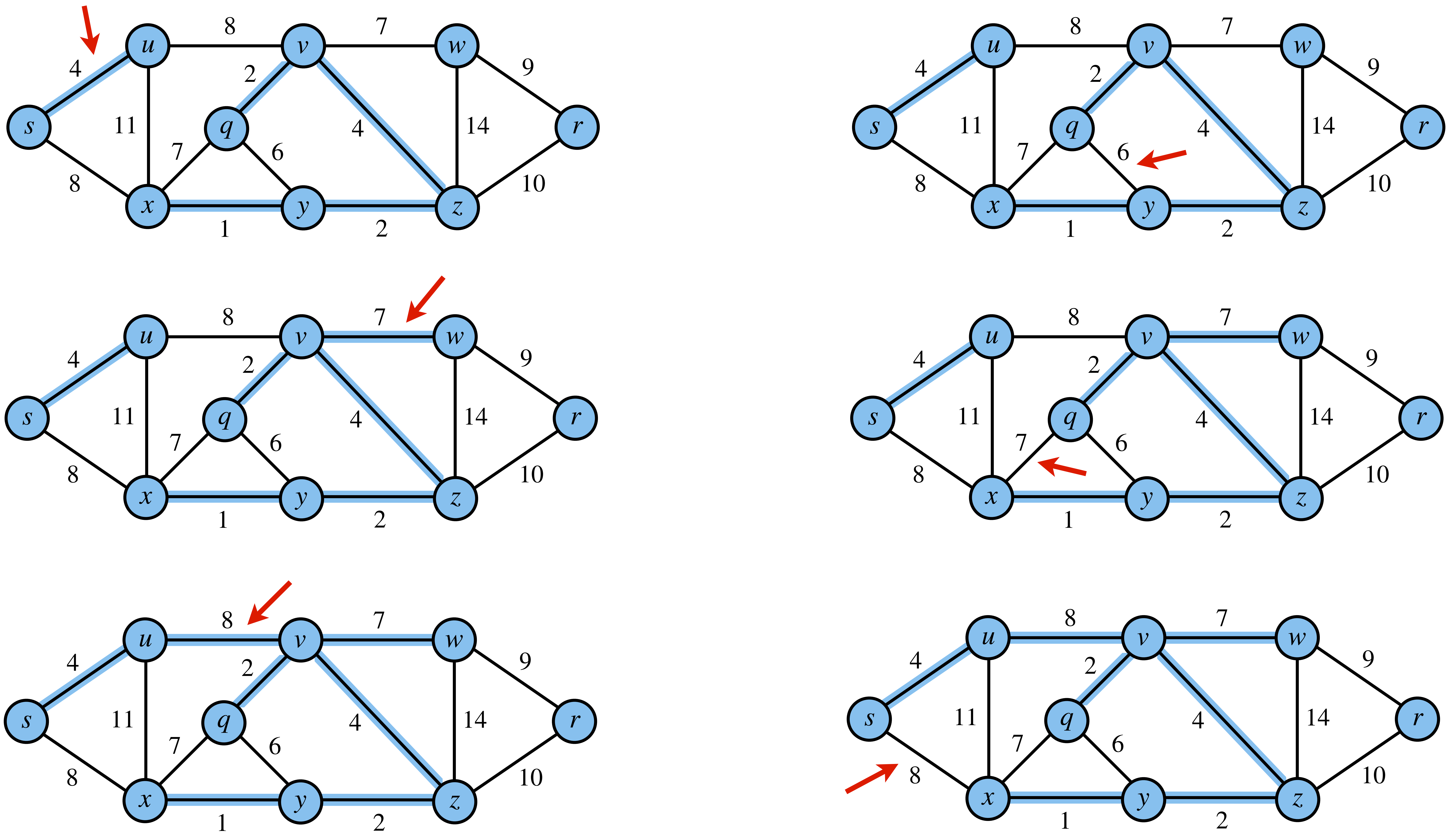

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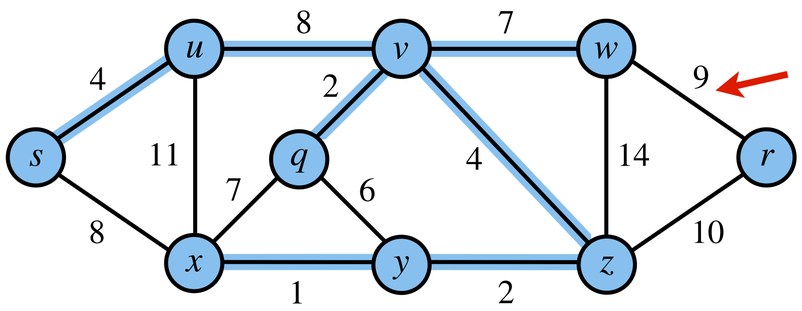

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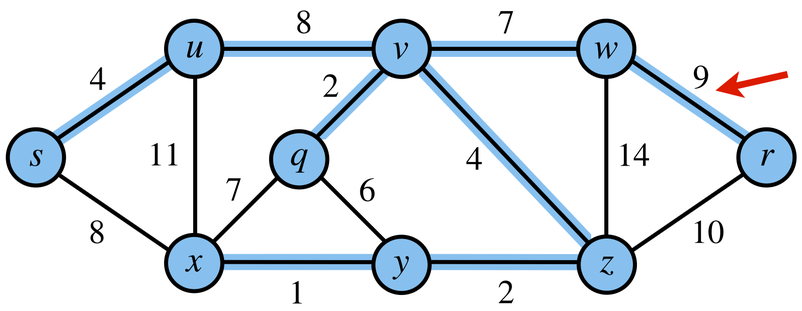

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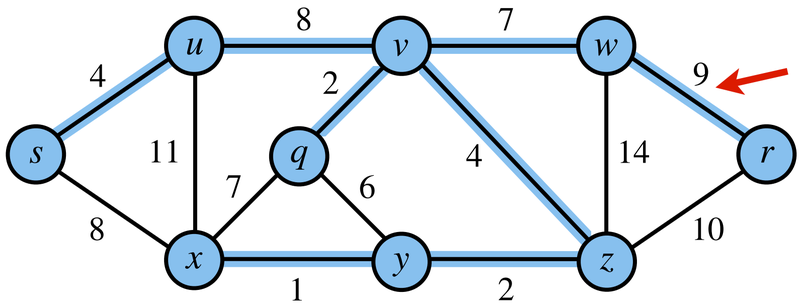

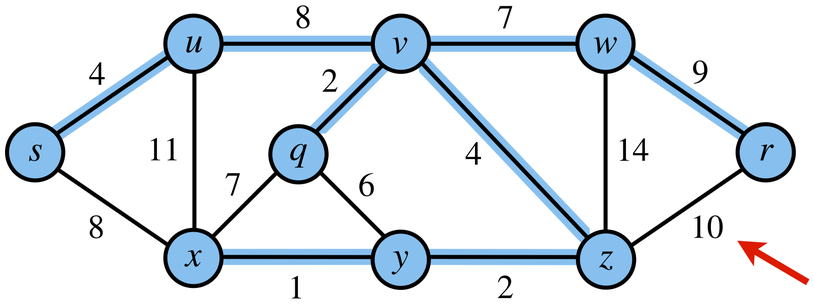

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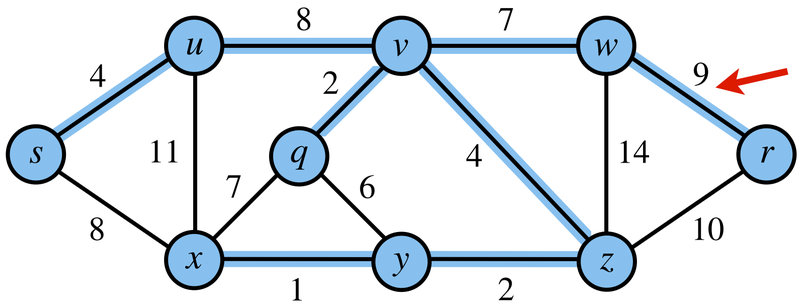

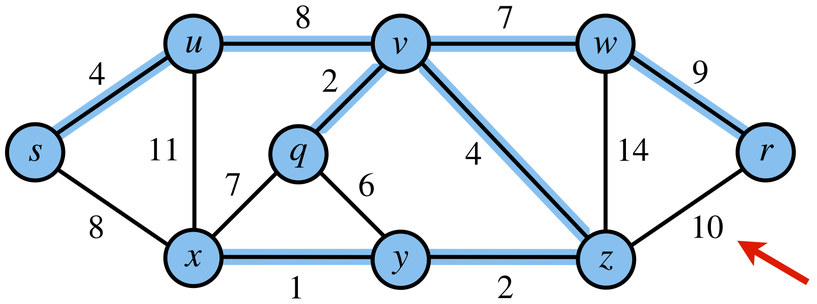

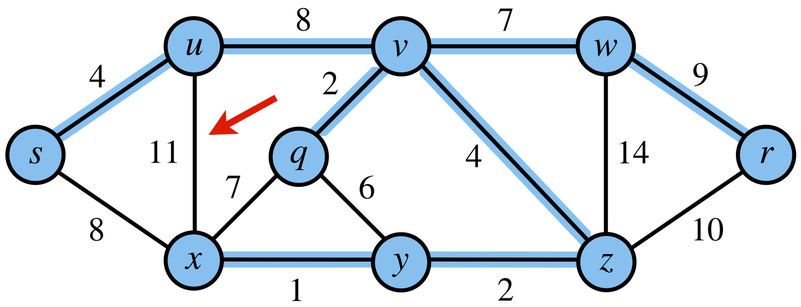

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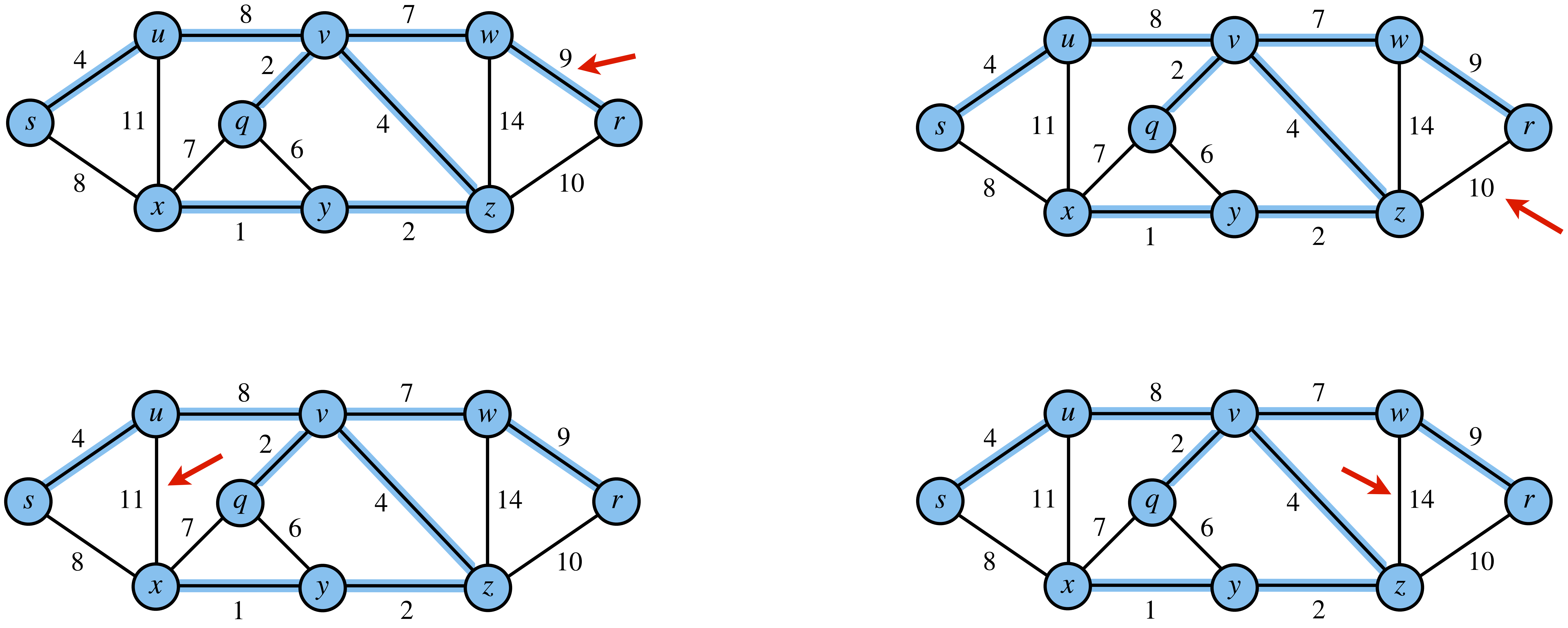

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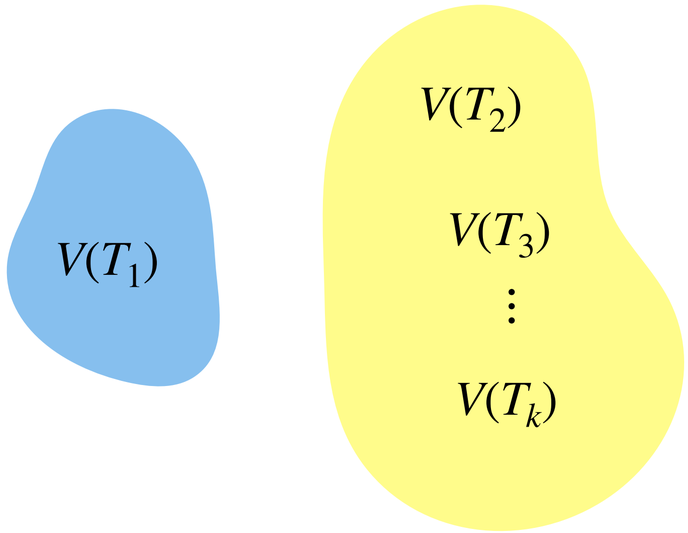

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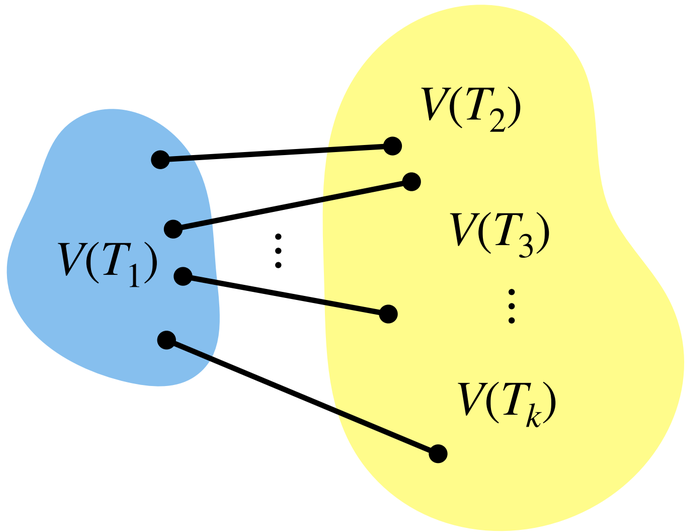

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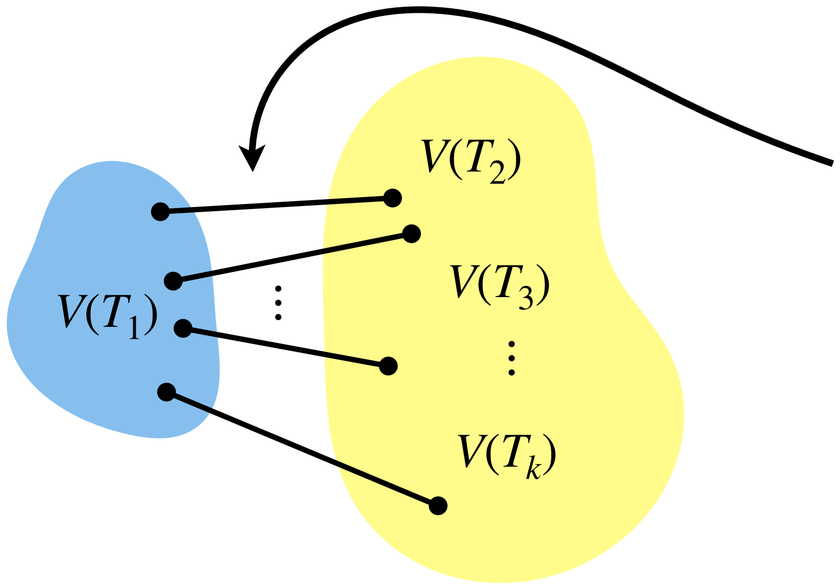

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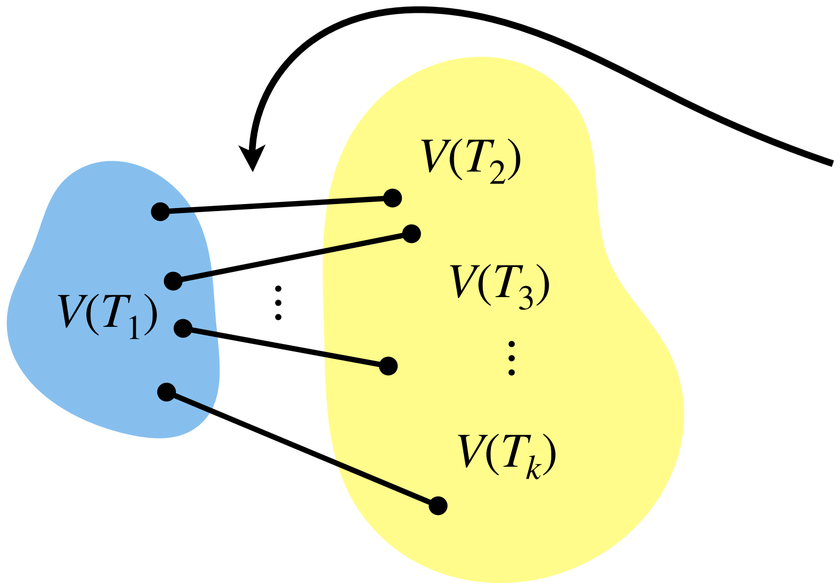

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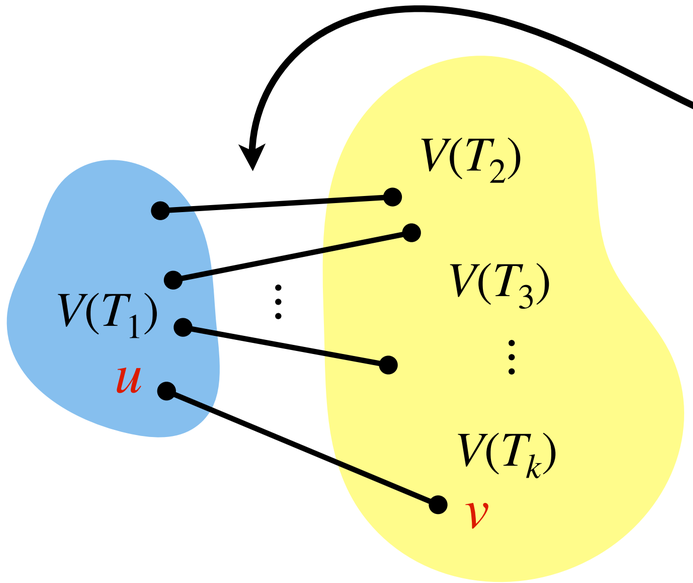

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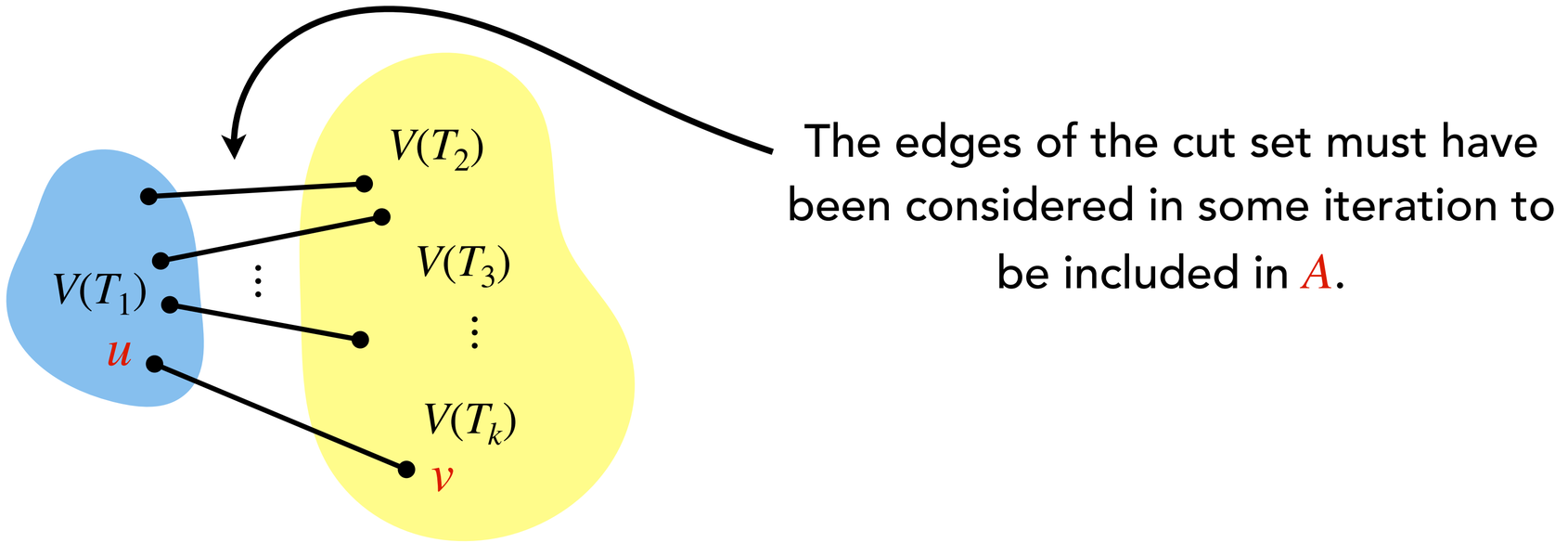

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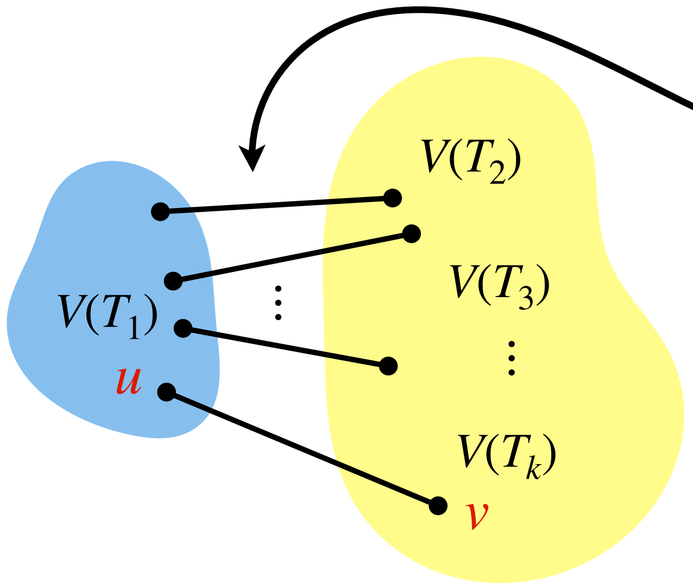

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Lemma: Let $G(V, E, w)$ an undirected, weighted and connected graph. Let A be a subset of E that is included in some MST of G. Let $C = (S, T)$ be any cut in G so that cut set of C does not have any edge from A. If e is the least weight edge in the cut-set of C, then $A \cup \{e\}$ is also part of some MST of G.

Recall this lemma.

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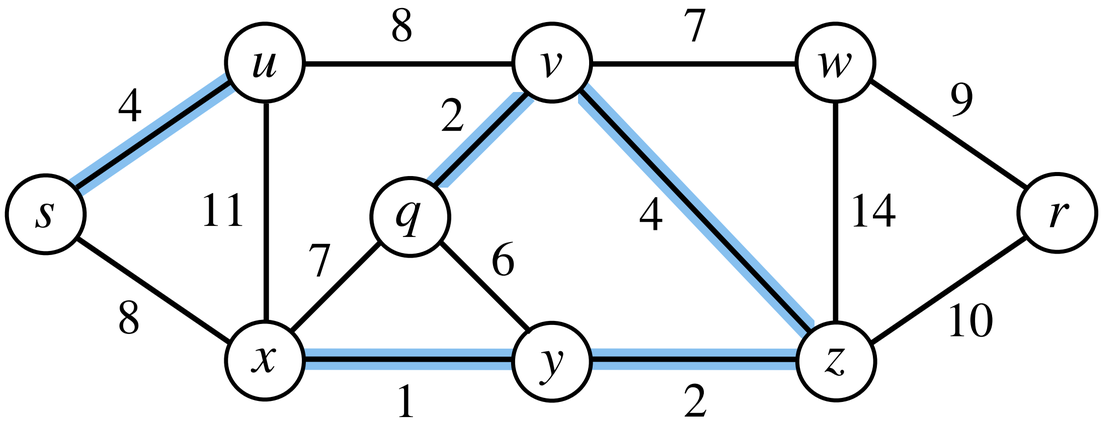

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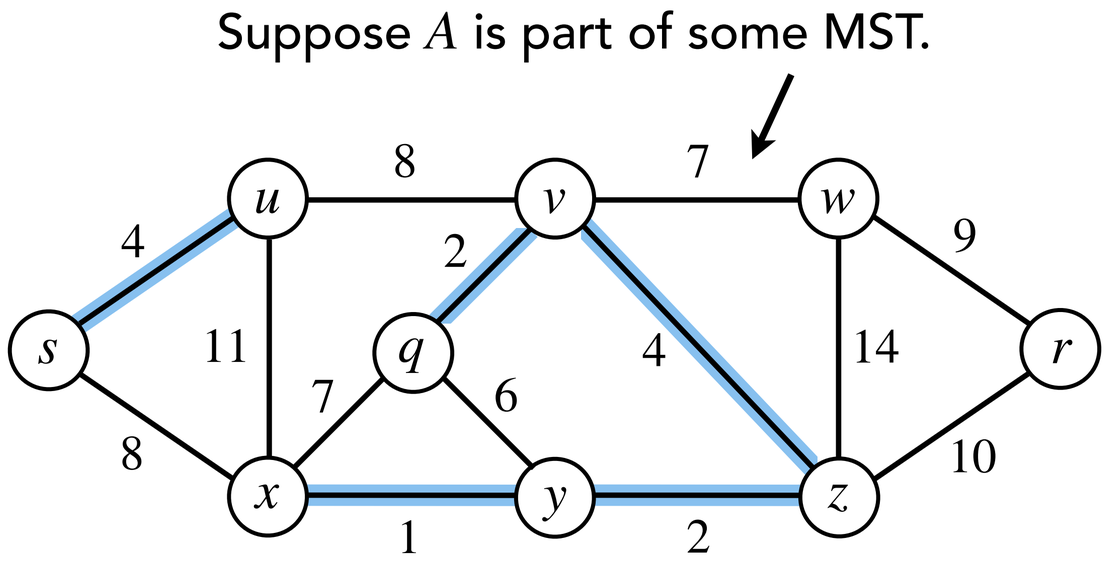

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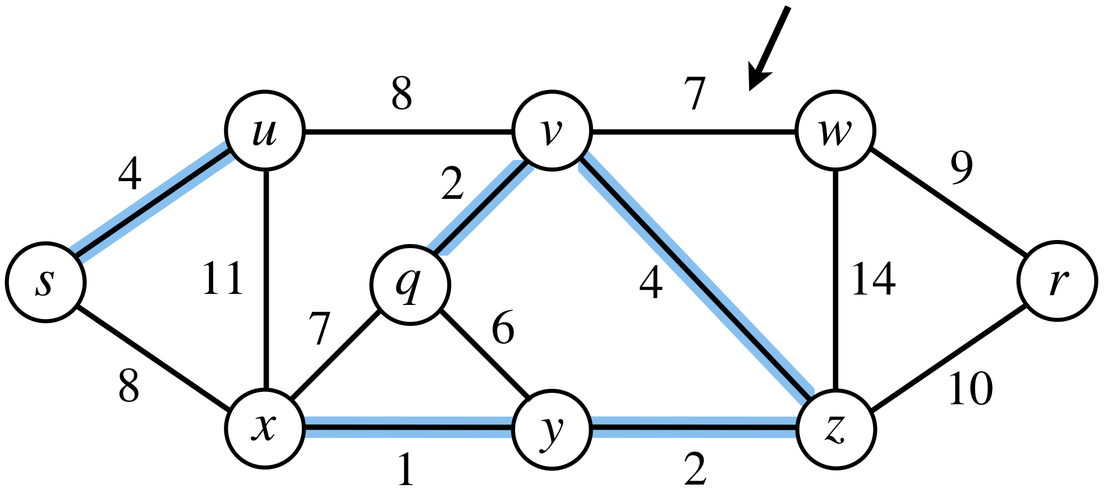

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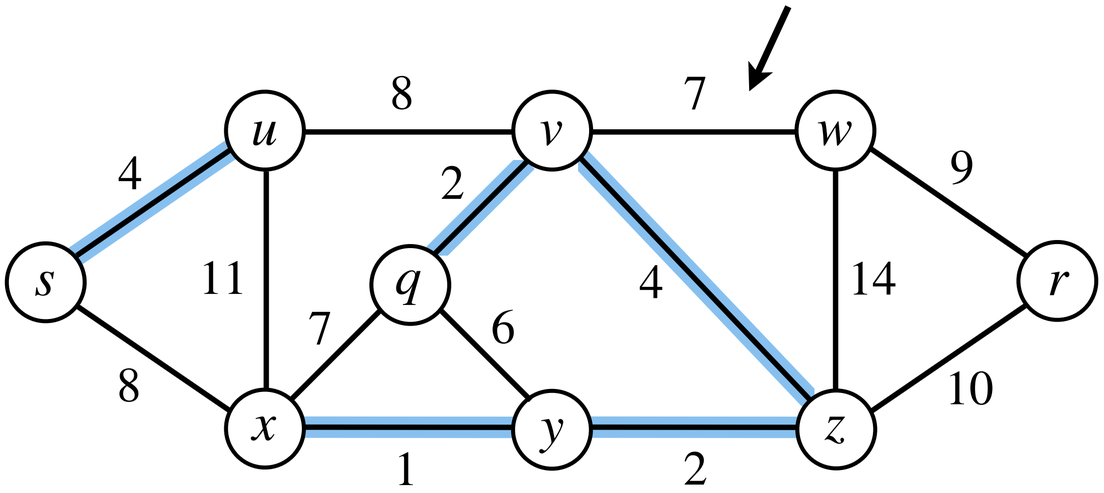

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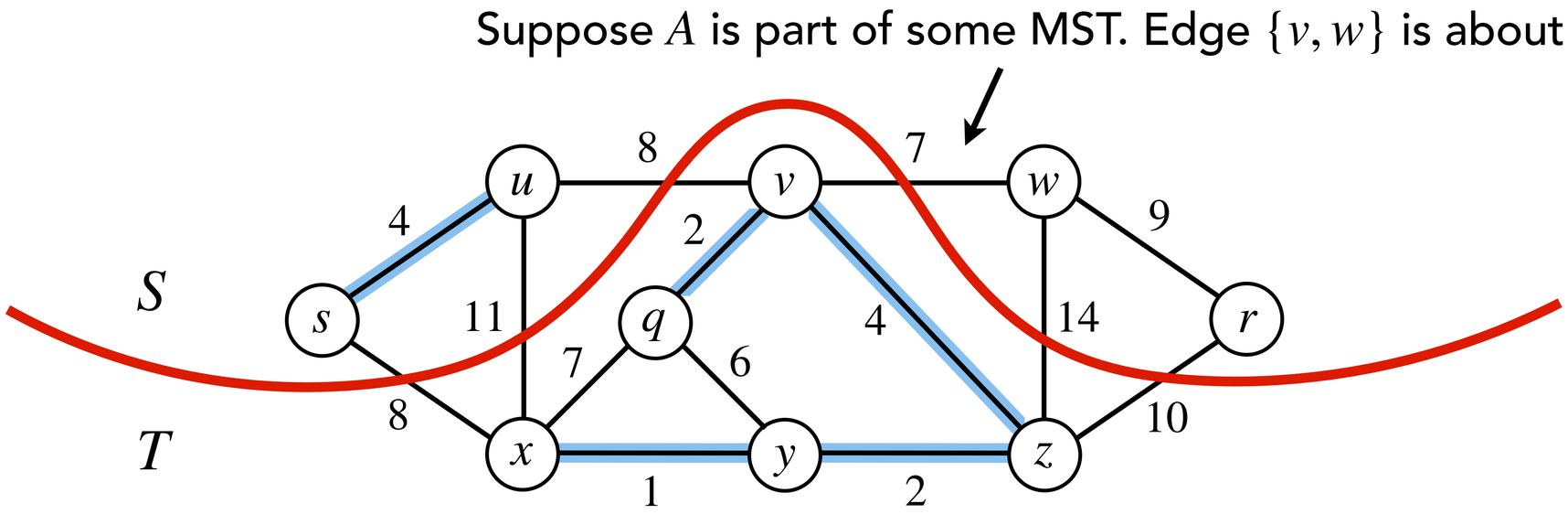

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